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# Open $G_2$ strings

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Abstract: We consider an open string version of the topological twist previously proposed for sigma-models with $G_2$ target spaces. We determine the cohomology of open strings states and relate these to geometric deformations of calibrated submanifolds and to flat or anti-self-dual connections on such submanifolds. On associative three-cycles we show that the worldvolume theory is a gauge-fixed Chern-Simons theory coupled to normal deformations of the cycle. For coassociative four-cycles we find a functional that extremizes on anti-self-dual gauge fields. A brane wrapping the whole $G_2$ induces a seven-dimensional associative Chern-Simons theory on the manifold. This theory has already been proposed by Donaldson and Thomas as the higher-dimensional generalization of real Chern-Simons theory. When the $G_2$ manifold has the structure of a Calabi-Yau times a circle, these theories reduce to a combination of the open A-model on special Lagrangians and the open B+$\bar{\text{B}}$-model on holomorphic submanifolds. We also comment on possible applications of our results.

Keywords: Sigma Models, Topological Field Theories, Topological Strings, Differential and Algebraic Geometry.

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## Contents

## 1. Introduction

Topological strings have been studied quite intensively as a toy model of ordinary string theory. Besides displaying a rich mathematical structure, they partially or completely control certain BPS quantities in ordinary string theory, and as such have found applications e.g. in the study BPS black holes and non-perturbative contributions to superpotentials.

Unfortunately, a full non-perturbative definition of topological string theory is still lacking, but it is clear that it will involve ingredients from both the A- and B-model, and that both open and closed topological strings will play a role. Since M-theory is crucial in understanding the strong coupling limit and nonperturbative properties of string theory, one may wonder whether something similar is true in the topological case, i.e. does there exist a seven-dimensional topological theory which reduces to topological string theory in six dimensions when compactified on a circle? And could such a seven-dimensional theory shed light on the non-perturbative properties of topological string theory?

In order to find such a seven-dimensional theory one can use various strategies. One can try to directly guess the spacetime theory, as in [1, 2], or one can try to construct a topological membrane theory as in [3 – 7] (after all, M-theory appears to be a theory of membranes, though the precise meaning of this sentence remains opaque). In this paper we will follow a different approach and study a topological version of strings propagating on a manifold of $G_2$ holonomy, following [8] (for an earlier work on $G_2$ sigma-models see [9]). In [8] the topological twist was defined using the extended worldsheet algebra that sigma-models on manifolds with exceptional holonomy possess [10]. For manifolds of

$G_2$ holonomy the extended worldsheet algebra contains the $c = 7/10$ superconformal algebra [9] that describes the tricritical Ising model, and the conformal block structure of this theory was crucial in defining the twist. In [8] it was furthermore shown that the BRST cohomology of the topological $G_2$ string is equivalent to the ordinary de Rham cohomology of the seven-manifold, and that the genus zero three-point functions are the third derivatives of a suitable prepotential, which turned out to be equal to the seven-dimensional Hitchin functional of [11]. The latter also features prominently in [1, 2], suggesting a close connection between the spacetime and worldsheet approaches.

In the present paper we will study open topological strings on seven-manifolds of $G_2$ holonomy, using the same twist as in [8]. There are several motivations to do this. First of all, we hope that this formalism will eventually lead to a better understanding of the open topological string in six dimensions. Second, some of the results may be relevant for the study of realistic compactifications of M-theory on manifolds of $G_2$ holonomy,[1] for a recent discussion of the latter see e.g. [12]. Third, by studying branes wrapping three-cycles we may establish a connection between topological strings and topological membranes in seven dimensions. And finally, for open topological strings one can completely determine the corresponding open string field theory [13], from which one can compute arbitrary higher genus partition functions and from which one can also extract highly non-trivial all-order results for the closed topological string using geometric transitions [14]. Repeating such an analysis in the $G_2$ case would allow us to use open $G_2$ string field theory to perform computations at higher genus in both the open and closed topological $G_2$ string. This is of special importance since the definition and existence of the topological twist at higher genus has not yet been rigorously established in the $G_2$ case.

Along the way we will run into various interesting mathematical structures and topological field theories in various dimensions that may be of interest in their own right.

The outline and summary of this paper is as follows. We will first briefly review the closed topological $G_2$ string and its Hilbert space. We will then consider open topological strings and their boundary conditions. Consistent boundary conditions are those which preserve one copy of the non-linear $G_2$ worldsheet algebra and were previously analyzed in [15, 16]. One finds that there are topological zero-, three-, four- and seven-branes in the theory.[2] The three- and four-branes wrap associative and coassociative cycles respectively and are calibrated by the covariantly constant three-form and its Hodge-dual which define the $G_2$ structure.

Next, we compute the topological open string spectrum in the presence of these branes. For a seven-brane, the spectrum has a simple geometric interpretation in terms of the Dolbeault cohomology of the $G_2$ manifold. To define the Dolbeault cohomology, we need to use the fact that $G_2 \subset \mathrm{SO}(7)$ acts naturally on differential forms, and we can decompose them into $G_2$ representations. In this paper, we will use the notation $\pi^p_{\mathbf{n}}$ to denote the projection of the space of $p$-forms $\Lambda^p$ onto the irreducible representation $\mathbf{n}$ of $G_2$. The

[1] This will require an extension of our results to singular manifolds which is an interesting direction for future research.

[2] It is unclear to us how we could incorporate coisotropic six-branes in our theory, whose existence is suggested in [17].

Dolbeault complex is then

$$0 \longrightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{\pi^2_{\mathbf{7}} d} \pi^2_{\mathbf{7}}(\Lambda^2) \xrightarrow{\pi^3_{\mathbf{1}} d} \pi^3_{\mathbf{1}}(\Lambda^3) \longrightarrow 0 \, . \tag{1.1}$$

The topological open string BRST cohomology is the cohomology of this complex and yields states at ghost numbers $0, 1, 2, 3$. For zero-, three- and four-branes the cohomology is obtained by reducing the above complex to the brane in question.

In section 4 we will verify explicitly that the BRST cohomology in ghost number one contains the space of (generalized) flat connections on the brane, but also contains the infinitesimal moduli of the topological brane. In particular, we will see that the topological open string reproduces precisely the results in the mathematics literature [18] regarding deformations of calibrated cycles in manifolds of $G_2$ holonomy.

We briefly discuss scattering amplitudes in section 5 and use them to construct the open topological string field theory following methods discussed in [13] in section 6. The final answer for the open topological string field theory turns out to be very simple. For seven-branes we obtain the associative Chern-Simons (CS) action

$$S = \int_Y *\phi \wedge CS_3(A) \, , \tag{1.2}$$

with $CS_3(A)$ the standard Chern-Simons three-form and $*\phi$ the harmonic four-form on the $G_2$ manifold $Y$. For the other branes we obtain the dimensional reduction of this action to the appropriate brane. The action (1.2) was first considered in [19, 20], and it is gratifying to have a direct derivation of this action from string theory. We will also discuss the dimensional reduction of this theory on $CY_3 \times S^1$, which leads to various real versions of the open A- and B-model, depending on the brane one is looking at. The situation is very similar to the closed topological $G_2$ string, which also reduced to a combination of real versions of the A- and B-models. It is presently unclear to us whether we should interpret this as meaning that the partition functions of the open and closed topological $G_2$ strings should not be interpreted as wave functions, as opposed to the partition functions of the open and closed A- and B-models, which are most naturally viewed as wave functions.

The last subject we discuss in section 6 is the emergence of worldsheet instanton contributions of the topological string theory on Calabi-Yau manifolds from the topological $G_2$ string on $CY_3 \times S^1$. Though our analysis is not yet conclusive, it appears that these worldsheet instanton effects arise from wrapped branes in the $G_2$ theory and not directly from worldsheet instantons.

Finally, in section 7 we make a preliminary investigation of the gauge-fixing and quantization of (1.2) and its reductions to four- and three-dimensional branes. As was the case in the open string field theory for A-branes, the gauge-fixed actions look very similar to the action (1.2) once we replace the ghost number one field $A$ by a field of arbitrary ghost number. We also study the one-loop partition functions of the various open string field theories, and find that they tend to have the effect of shifting the tree-level theories in a rather simple way. This is similar to the one-loop shift $k \to k + h(G)$ of the level $k$ in ordinary Chern-Simons theory, with $h(G)$ the dual Coxeter number of the gauge group $G$.

In particular, we find that $*\phi$ in (1.2) is shifted by a four-form proportional to the first Pontrjagin class of the manifold $Y$. We have not yet attempted to determine whether (1.2) is renormalizable and well-defined as a quantum theory (which, by naive power-counting, it is not) but we expect that it should be as it is equivalent to a string theory (a similar issue occurs for holomorphic Chern-Simons in the B-model).

We conclude with a list of open problems and have collected various technical results in the appendices.

We will adhere to the following conventions: $M$ will refer to a calibrated submanifold of dimension 3 or 4 (i.e. calibrated by $\phi$ or $*\phi$, respectively); these are known, respectively, as associative and coassociative submanifolds. The ambient $G_2$ manifold will be denoted $Y$.

## 2. A brief review of the closed topological $G_2$ string

Let us briefly review the definition of the topological $G_2$ string found in [8]. We will cover only essential points. For further details we refer the reader to [8].

### 2.1 Sigma model for the $G_2$ string

The topological $G_2$ string constructs a topological string theory with target space a seven-dimensional $G_2$-holonomy manifold $Y$. This topological string theory is defined in terms of a topological twist of the relevant sigma-model. In order to have $\mathcal{N} = 1$ target space supersymmetry, one starts with an $\mathcal{N} = (1,1)$ sigma model on a $G_2$ holonomy manifold. The special holonomy of the target space implies an extended supersymmetry algebra for the worldsheet sigma-model [10]. That is, additional conserved supercurrents are generated by pulling back the covariantly constant 3-form $\phi$ and its hodge dual $*\phi$ to the worldsheet as

$$\phi_{\mu\nu\rho}(\mathbf{X})D\mathbf{X}^{\mu}D\mathbf{X}^{\nu}D\mathbf{X}^{\rho} \ ,$$

where $\mathbf{X}$ is a worldsheet chiral superfield, whose bosonic component corresponds to the world-sheet embedding map. From the classical theory it is then postulated that the extended symmetry algebra survives quantization, and is present in the quantum theory. This postulate is also based on analyzing all possible quantum extensions of the symmetry algebra compatible with spacetime supersymmetry and $G_2$ holonomy.

A crucial property of the extended symmetry algebra is that it contains an $\mathcal{N} = 1$ SCFT sub-algebra, which has the correct central charge of c=7/10 to correspond to the tri-critical Ising unitary minimal model. Unitary minimal models have central charges in the series $c = 1 - \frac{6}{p(p+1)}$ (for $p$ an integer) so the tri-critical Ising model has $p = 4$.

The conformal primaries for such models are labelled by two integer Kac labels, $n'$ and $n$, as $\phi_{(n',n)}$ where $1 \leq n' \leq p$ and $1 \leq n < p$. The Kac labels determine the conformal weight of the state as $h_{n',n} = \frac{[pn'-(p+1)n]^2-1}{4p(p+1)}$. The Kac table for this minimal model is reproduced in [8, table 1]. Note that primaries with label $(n', n)$ and $(p+1-n', p-n)$ are equivalent. This model has six conformal primaries with weights $h_I = 0, 1/10, 6/10, 3/2$ (for the NS states) and $h_I = 7/16, 3/80$ (for the R states).



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The conformal block structure of the weight 1/10, $\phi_{(2,1)}$, and of the weight 7/16 primary, $\phi_{(1,2)}$, is particularly simple,

$$
\begin{aligned}
\phi_{(2,1)} \times \phi_{(n',n)} &= \phi_{(n'-1,n)} + \phi_{(n'+1,n)} \ , \\
\phi_{(1,2)} \times \phi_{(n',n)} &= \phi_{(n',n-1)} + \phi_{(n',n+1)} \ ,
\end{aligned}
$$

where $\phi_{(n',n)}$ is any primary. This conformal block decomposition is schematically denoted as

$$
\begin{aligned}
\Phi_{(2,1)} &= \Phi^{\downarrow}_{(2,1)} \oplus \Phi^{\uparrow}_{(2,1)} \ , \\
\Phi_{(1,2)} &= \Phi^{-}_{(1,2)} \oplus \Phi^{+}_{(1,2)} \ .
\end{aligned} \tag{2.1}
$$

The conformal primaries of the full sigma-model are labelled by their tri-critical Ising model highest weight, $h_I$, and the highest weight corresponding to the rest of the algebra, $h_r$, as $|h_I, h_r\rangle$. This is possible because the stress tensors, $T_I$, of the tricritical sub-algebra and of the 'rest' of the algebra, $T_r = T - T_I$ (where $T$ is the stress tensor of the full algebra), satisfy $T_I \cdot T_r \sim 0$.

### 2.2 The $G_2$ twist

The standard $\mathcal{N} = (2,2)$ sigma-models can be twisted by making use of the U(1) R-symmetry of their algebra. Using the U(1) symmetry, the twisting can be regarded as changing the worldsheet sigma-model with a Calabi-Yau target space by the addition of the following term:

$$
\pm \frac{\omega}{2} \overline{\psi} \psi \ , \tag{2.2}
$$

with $\omega$ the spin connection on the world-sheet. This effectively changes the charge of the fermions under worldsheet gravity to be integral, resulting in the topological A/B-model depending on the relative sign of the twist in the left and right sector of the theory (for fermions with holomorphic or anti-holomorphic target space indices). Here $\overline{\psi}$ and $\psi$ can be either left- or right-moving worldsheet fermions and $\omega$ is the spin-connection on the worldsheet. In the topological theory, before coupling to gravity, there are no ghosts or anti-ghosts so these are the only spinors/fermions in the system.

This twist has been re-interpreted [21, 22] as follows. First think of the exponentiation of (2.2) as an insertion in the path integral rather than a modification of the action. By bosonising the world-sheet fermions we can write $\overline{\psi}\psi = \partial H$ for a free boson field so the above becomes

$$
\int \frac{\omega}{2} \partial H = - \int H \frac{\partial \omega}{2} = \int H R \ , \tag{2.3}
$$

where $R$ is the curvature of the world-sheet. We can always choose a gauge for the metric such that $R$ will only have support on a number of points given by the Euler number of the worldsheet.

For closed strings on a sphere the Euler class has support on two points which can be chosen to be at 0 and $\infty$ (in the CFT defined on the sphere) so the correlation functions in the topological theory can be calculated in terms of the original CFT using the following dictionary:

$$\langle \ldots \rangle_{\text{twisted}} = \left\langle e^{H(\infty)} \ldots e^{H(0)} \right\rangle_{\text{untwisted}} \ . \tag{2.4}$$

The 'untwisted' theory should not be confused with the physical theory, because it does not include integration over world-sheet metrics and hence has no ghost or superghost system and also it is still not at the critical dimension. The equation above simply relates the original untwisted $\mathcal{N}=2$ sigma-model theory to the twisted one.

In [8] a related prescription is given to define the twisted 'topological' sigma-model on a 7-dimensional target space with $G_2$ holonomy. Here the role of the U(1) R-symmetry is played by the tri-critical Ising model sub-algebra. However, a difference is that the topological $G_2$ sigma-model is formulated in terms of conformal blocks rather than in terms of local operators. In particular the operator $H$ in the above is replaced by the conformal block $\Phi^{+}_{(1,2)}$.

The main point of the topological twisting is to redefine the theory in such a way that it contains a scalar BRST operator. In the $G_2$ sigma model, the BRST operator is related to the conformal block of the weight 3/2 current $G(z)$ of the super stress-energy tensor,[3]

$$Q = G^{\downarrow}_{-\frac{1}{2}} \ .$$

It should be pointed out that in [8] it was not possible to explicitly construct the twisted stress tensor, and although there is circumstantial evidence that the topological theory does exist beyond tree level this statement remains conjectural.

### 2.3 The $G_2$ string Hilbert space

In a general CFT the set of states can be generated by acting with primary operators and their decendants on the vacuum state, resulting in an infinite dimensional Fock space. In string sigma models this Fock space contains unphysical states, and so the physical Hilbert space is given by the cohomology of the BRST operator on this physical Hilbert space which is still generally infinite-dimensional.

In the topological A- and B-models a localization argument [22] implies that only BRST fixed-points contribute to the path integral and these correspond to holomorphic and constant maps, respectively. Thus the set of field configurations that when quantized, generate states in the Hilbert space is restricted to this subclass of all field configurations and so the Fock space is much smaller. Upon passing to BRST cohomology this space actually becomes finite-dimensional.

In the $G_2$ string the localization argument cannot be made rigorous, because the action of the BRST operator on the worldsheet fields is inherently quantum, and so is not well defined on the classical fields. Neglecting this issue and proceeding naively, however, one can construct a localization argument for $G_2$ strings that suggests that the path integral

[3]The super stress-energy tensor is given as $\mathbf{T}(z,\theta) = G(z) + \theta T(z)$. The current G(z) can be further decomposed as $G(z) = \Phi_{(2,1)} \otimes \Psi_{\frac{14}{10}}$, in terms of the tri-critical Ising-model part and the rest of the algebra, respectively. Since its tri-critical Ising model part contains only the primary $\Phi_{(2,1)}$, it can be decomposed into conformal blocks accordingly.



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localizes on the space of constant maps [8]. Thus we will take our initial Hilbert space to consist of states generated by constant modes $X^\mu_0$ and $\psi^\mu_0$ on the world-sheet (in the NS-sector there is no constant fermionic mode but the lowest energy mode $\psi^\mu_{-\frac{1}{2}}$ is used instead). These correspond to solutions of worldsheet equations of motion with minimal action which dominate the path integral in the large volume limit.

In [22] the fact that the path integral can be evaluated by restricting to the space of BRST fixed points is related to another feature of the A/B-models: namely the coupling-invariance (modulo topological terms) of the worldsheet path integral. Variations of the path integral with respect to the inverse string coupling constant $t \propto (\alpha')^{-1}$ are $Q$-exact, so one may freely take the weak coupling limit $t \to \infty$ in which the classical configurations dominate. This limit is equivalent to rescaling the target space metric, and so we will refer to it as the large volume limit.

Accordingly, all the calculations in the A- and B- model can be performed in the limit where the Calabi-Yau space has a large volume relative to the string scale, and the worldsheet theory can be approximated by a free theory. The $G_2$ string also has the characteristics of a topological theory, such as correlators being independent of the operator's positions, and the fact that the BRST cohomology corresponds to chiral primaries. On the other hand since the theory is defined in terms of the conformal blocks, it is difficult to explicitly check the coupling constant independence. Based on the topological arguments, and on the postulate of the quantum symmetry algebra, in this paper we will assume the coupling constant independence and the validity of localization arguments. Even if these arguments should fail for subtle reasons, the results of this paper are always valid in the large volume limit.

### 2.4 The $G_2$ string and geometry

As in the topological A- and B-model, for the topological $G_2$ string there is a one-to-one correspondence between local operators of the form $O_{\omega_p} = \omega_{i_1\ldots i_p}\psi^{i_1}\ldots\psi^{i_p}$ and target space $p$-forms $\omega_p = \omega_{i_1\ldots i_p} dx^{i_1}\wedge\ldots\wedge dx^{i_p}$. In [8] it is found that the BRST cohomology of the left (right) sector alone maps to a certain refinement of the de Rham cohomology described by the '$G_2$ Dolbeault' complex

$$0 \to \Lambda^0_{\mathbf{1}} \xrightarrow{\check{D}} \Lambda^1_{\mathbf{7}} \xrightarrow{\check{D}} \Lambda^2_{\mathbf{7}} \xrightarrow{\check{D}} \Lambda^3_{\mathbf{1}} \to 0\ . \tag{2.5}$$

The notation is that $\Lambda^p_{\mathbf{n}}$ denotes differential forms of degree $p$, transforming in the irreducible representation $\mathbf{n}$ of $G_2$. The operator $\check{D}$ acts as the exterior derivative on 0-forms, and as

$$\begin{aligned} \check{D}(\alpha) &= \pi^2_{\mathbf{7}}(d\alpha) \quad \text{if} \quad \alpha\in\Lambda^1\ , \\ \check{D}(\beta) &= \pi^3_{\mathbf{1}}(d\beta) \quad \text{if} \quad \beta\in\Lambda^2\ , \end{aligned}$$

where $\pi^2_{\mathbf{7}}$ and $\pi^3_{\mathbf{1}}$ are projectors onto the relevant representations. The explicit expressions for the projectors and the standard decomposition of the de Rham cohomology are included in appendix A. Thus, the BRST operator $G^{\downarrow}_{-1/2}$ maps to the differential operator of the complex $\check{D}$. In the closed theory, combining the left- and right-movers, one obtains the full cohomology of the target manifold, accounting for all geometric moduli: metric deformations, the $B$-field moduli, and rescaling of the associative 3-form $\phi$. The relevant cohomology for the open string states will be worked out in the following sections.

## 3. Open string cohomology

We will now consider the $Q$ cohomology of the open string states. Later, we will interpret part of this cohomology in terms of geometric and non-geometric (gauge field) moduli on calibrated 3- and 4-cycles.

In [8] states in the $G_2$ CFT were shown to satisfy a certain non-linear bound in terms of $h_I$ and $h_r$ and states saturating this bound are argued to fall into shorter, BPS, representation of the non-linear $G_2$ operator algebra. Such states are referred to as chiral primaries. Analogous to the $\mathcal{N}=2$ case, it is the physics of these primaries that the twist is intended to capture and thus they are the states that occur in the BRST cohomology. The chiral primaries in the NS sector have $h = 0, 1/2, 1, 3/2$ and $h_I = 0, 1/10, 6/10, 3/2$ and they are the image of the RR ground states under spectral flow.

Recall that we are working in the zero mode approximation (corresponding to the large volume limit, $t \to \infty$, where oscillator modes can be neglected) and in this limit a general state is of the form $A_{\mu_1 \ldots \mu_n}(X_0)\psi_0^{\mu_1} \ldots \psi_0^{\mu_n}$. On such states $L_0$ acts as $t\Box + \frac{n}{2}$ so states with $h = 0, 1/2, 1, 3/2$ correspond to 0, 1, 2, and 3 forms ($f(X_0)$, $A_\mu(X_0)\psi_0^\mu$, $\ldots$). As argued in [8] we can thus consider $Q$-cohomology on the space of 0, 1, 2, and 3 forms restricted to those that have $h_I = 0, 1/10, 6/10, 3/2$, respectively.

In general we are interested in harmonic representatives of the $Q$ cohomology so we will look for operators (corresponding to states) that are both $Q$- and $Q^\dagger$-closed. The results we obtain are essentially the same as those for one side of the closed worldsheet theory [8].

### 3.1 Degree one

We will start by looking at the $h = 1/2$ state, because it is the only one that will generate a marginal deformation of the theory. A general state with $h = 1/2$ is of the form $A_\mu(X)\psi^\mu$ so long as[4]

$$[L_0, A_\mu(X)] = t\Box A_\mu(X) = 0 \tag{3.1}$$

It also satisfies

$$[L_0^I, A_\mu(X)\psi^\mu] = \frac{1}{10} A_\mu(X)\psi^\mu \ ,$$

so it is a chiral primary (i.e. it saturates the chiral bound). Because it is a chiral primary, it has to be $Q$-closed [8]. Rather than proceed along these lines, however, we will consider the $Q$-cohomology directly from the definition of $Q$.

[4] Although we will sometimes use the full fields $X$ and $\psi$ in the CFT and also consider OPE's which generate deriviatives of these fields the reader should recall that we are always working in the large volume limit where these reduce to $X_0$ and $\psi_0$.

Let us determine the $Q$-cohomology of 1-forms $\mathcal{A} = A_\mu(X)\psi^\mu$. We first calculate $\{G_{-\frac{1}{2}}, A_\mu(X)\psi^\mu\}$ in the CFT on the complex plane with $z$ complex 'bulk' coordinates and $y$ 'boundary' coordinates on the real line

$$\begin{aligned}
\{G_{-1/2}, A_\mu(X)\psi^\mu\} &= \oint dz\, G(z) \cdot A_\mu(X)\psi^\mu(y) \;, \\
G(z) \cdot A_\mu(X)\psi^\mu(y) &= g_{\rho\sigma}(X)\psi^\rho \partial X^\sigma(z) \cdot A_\mu(X)\psi^\mu(y) \\
&\sim \partial(\ln|z-y|^2 + \ln|\overline{z}-y|^2)\nabla_\rho A_\mu \psi^\rho(z)\psi^\mu(y) \\
&\quad + \frac{1}{z-y}\partial X^\mu(z) A_\mu(X(y)) \;.
\end{aligned} \tag{3.2}$$

This gives[5]

$$\{G_{-1/2}, A_\mu(X)\psi^\mu\} = A_\mu \partial X^\mu(y) + \frac{1}{2}\partial_{[\mu}A_{\nu]}\psi^\mu\psi^\nu \;. \tag{3.3}$$

To compute the action of $Q$ we now project onto the $\downarrow$ part, which includes only the part with tri-critical Ising weight 6/10. The term $A_\mu \partial X^\mu$ vanishes in the zero mode limit so we only need to consider the second term. The condition that this term has $h_I = \frac{6}{10}$ is [8]

$$\left(\pi^2_{\mathbf{14}}\right)^{\rho\sigma}{}_{\mu\nu}\, \partial_{[\rho}A_{\sigma]} = 0 \;, \tag{3.4}$$

where $\pi^2_{\mathbf{14}}$ is the projector onto the 2-form subspace $\Lambda^2_{\mathbf{14}} \subset \Lambda^2$, in the $\mathbf{14}$ representation of $G_2$.

This result implies that the $\frac{6}{10}$ part of $\partial_{[\rho}A_{\sigma]}$ (or any 2-form) is in $\Lambda^2_{\mathbf{7}}$, so on a 1-form we can define $Q$ as

$$\{Q, A_\mu\psi^\mu\} = (\pi^2_{\mathbf{7}})\{G_{-\frac{1}{2}}, A_\mu\psi^\mu\} = 6\phi_{\mu\nu}{}^\gamma \phi_\gamma{}^{\rho\sigma}\partial_{[\rho}A_{\sigma]}dx^\mu \wedge dx^\nu = \check{D}A = 0 \;, \tag{3.5}$$

where we have used

$$(\pi^2_{\mathbf{7}})^{\rho\sigma}{}_{\mu\nu} = 4(*\phi)^{\rho\sigma}{}_{\mu\nu} + \frac{1}{6}(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\rho_\nu) = 6\phi_{\mu\nu}{}^\gamma\phi_\gamma{}^{\rho\sigma} \;. \tag{3.6}$$

Note that $Q$ acting on 1-forms has reduced essentially to $\check{D}$; the same will occur for forms of other degrees.

Let us now consider $Q$-coclosure. The inner product of states

$$\left\langle A_{[\mu\nu]}\psi^\mu\psi^\nu | B_{[\alpha\beta]}\psi^\alpha\psi^\beta \right\rangle \;,$$

becomes the inner product of forms $\int_Y (*A \wedge B)$, so $Q^\dagger$ acting on $A$ is given by $\langle Q \cdot f(X) | A_\mu(X)\psi^\mu\rangle = \langle f(X) | Q^\dagger \cdot A_\mu\psi^\mu\rangle$, which can be determined as

$$\langle Q \cdot f(X) | A_\mu(X)\psi^\mu\rangle = \int \sqrt{g}\partial_\mu f(X) A^\mu(X) = -\int \sqrt{g} f(X)\nabla_\mu A^\mu(X) \;. \tag{3.7}$$

So if $A_\mu$ is also required to satisfy

$$Q^\dagger \cdot A_\mu(X)\psi^\mu = -\nabla_\mu A^\mu(X) = 0 \;, \tag{3.8}$$

then it is $Q$- and $Q^\dagger$-closed and hence a harmonic represenative of $Q$-cohomology.

[5] We have not been careful about the relative normalizations of the bosonic and fermionic bulk-boundary OPE's, but this is not relevant as in all computations of this type that occur below, we will only end up keeping one of the terms.

### 3.2 Degree zero

The cohomology in degree zero is rather trivial. Given a degree zero mode $f(X)$ we have $\{Q, f(X)\} = \partial_\mu f(x)\psi^\mu$. This follows from $Q = G^{\downarrow}_{-\frac{1}{2}} = G_{-\frac{1}{2}}$, because the projection onto the $\downarrow$ component is trivial since all operators of the form $A_\mu(X)\psi^\mu$ automatically have $L_0^I$ weight $\frac{1}{10}$. So $Q$-closure implies

$$\partial_\mu f(X) = 0 \; . \tag{3.9}$$

The $Q^\dagger$-closure here is vacuous since there are no lower degree fields.

### 3.3 Degree two

In degree two we start with a two form $\omega_{\rho\sigma}\psi^\rho\psi^\sigma$ which should have $L_0^I$ weight $\frac{6}{10}$, so it should satisfy $\pi^2_{\mathbf{7}}(\omega) = \omega$. The need to restrict $\omega \in \Lambda^2_{\mathbf{7}}$ comes from the way $Q$ is defined in [8]. We must once more calculate the action of $G_{-\frac{1}{2}}$, and then project it onto the $\downarrow$ part

$$\begin{aligned}
\{G_{-\frac{1}{2}}, \omega\} &= \oint dz\; g_{\mu\nu}\psi^\mu \partial X^\nu(z) \cdot \omega_{\rho\sigma}\psi^\rho\psi^\sigma \\
&= \oint dz\; \frac{1}{z} g_{\mu\nu}\partial^\nu \omega_{\rho\sigma}\psi^\mu\psi^\rho\psi^\sigma + \frac{1}{z} g_{\mu\nu}\partial X^\nu \omega_{\rho\sigma} g^{\mu\rho}\psi^\sigma - \frac{1}{z} g_{\mu\nu}\partial X^\nu \omega_{\rho\sigma} g^{\mu\sigma}\psi^\rho \\
&= \partial_\mu \omega_{\rho\sigma}\psi^\mu\psi^\rho\psi^\sigma + 2\omega_{\rho\sigma}\partial X^\rho \psi^\sigma \; .
\end{aligned} \tag{3.10}$$

Once more we can drop the second term in the large volume limit in which we are working. We use the result in [8] that the projector onto the $L_0^I$ weight $\frac{3}{2}$ corresponds to the projector onto $\Lambda^3_{\mathbf{1}}$, and is given by contracting with the associative 3-form $\phi$. So for $\Omega \in \Lambda^3_{\mathbf{1}}$

$$\phi^{\alpha\beta\gamma}\Omega_{\alpha\beta\gamma}\phi_{\mu\nu\rho} = 7\Omega_{\mu\nu\rho} \; . \tag{3.11}$$

In particular, we can project onto the $\frac{3}{2}$ part of $\{G_{-\frac{1}{2}}, \omega\} = \partial_\mu \omega_{\rho\sigma}$ using $\phi^{\alpha\beta\gamma}$, so $Q$-closure implies

$$\phi^{\alpha\beta\gamma}\partial_{[\alpha}\omega_{\beta\gamma]} = 0 \; . \tag{3.12}$$

Note that this once again can be written as $\check{D}\omega = 0$.

We will now derive the $Q^\dagger$-closure condition. This is done in exactly the same way as was done for the degree one components

$$\langle \omega | Q \cdot A_\mu(X)\psi^\mu \rangle = \int \sqrt{g}\omega^{\mu\nu}(\pi^2_{\mathbf{7}})^{\alpha\beta}{}_{\mu\nu}\partial_\alpha A_\beta = -\int \sqrt{g} A_\beta \left( (\pi^2_{\mathbf{7}})^{\alpha\beta}{}_{\mu\nu} \nabla_\alpha \omega^{\mu\nu} \right) , \tag{3.13}$$

so

$$Q^\dagger \cdot \omega = -(\pi^2_{\mathbf{7}})^{\mu\nu}{}_{\alpha\beta}\nabla^\alpha \omega_{\mu\nu} dx^\beta = -6\phi^{\mu\nu}{}_\gamma \phi^\gamma{}_{\alpha\beta}\nabla^\alpha\omega_{\mu\nu}dx^\beta = -\nabla^\alpha \omega_{\alpha\beta}dx^\beta = 0 \; . \tag{3.14}$$

Here we have used $\pi^2_{\mathbf{7}}(\omega) = \omega$.



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### 3.4 Degree three

A 3-form $\Omega_{\mu\nu\rho}\psi^\mu\psi^\nu\psi^\rho$ is first projected onto its $\Lambda^3_{\mathbf{1}}$ component by $Q$, so we take $\pi^3_{\mathbf{1}}(\Omega) = \Omega$, which means that $\Omega$ is a function times $\phi$. From the definition of $Q$ it is evident that it acts trivially on $\Omega$ since there is no higher $L_0^I$ eigenstate in the NS sector for $Q$ to project onto. This implies $Q = 0$ on three forms which matches (2.5). Thus we see that the action of $Q$ on states in the zero mode approximation maps into the complex (2.5) as anticipated in section 2.4.

The $Q$-coclosure of $\Omega$ is derived similarly to the 1- and 2-form case and gives

$$Q^\dagger \cdot \Omega = \nabla^\mu \Omega_{\mu\nu\rho} dx^\nu \wedge dx^\rho = 0 \; . \tag{3.15}$$

### 3.5 Harmonic constraints

In the previous subsections we considered the conditions for $Q$- and $Q^\dagger$-closure on the states in the $G_2$ CFT. These conditions are all linear in derivatives but they must be enforced simultaneously to generate unique representatives of $Q$-cohomology. As $Q$ corresponds to the operator $\check{D}$ discussed in section 2.4, it generates the Dolbeault complex (2.5) which is known to be elliptic [23, 19] and so can be studied using Hodge theory. This implies that physical states in the theory correspond to the kernel of the Laplacian operator $\{Q, Q^\dagger\}$, so one can equivalently consider this single non-linear condition instead of the two seperate linear conditions imposed by $Q$ and $Q^\dagger$.

These $Q$-harmonic conditions (derived from the actions of $Q$ and $Q^\dagger$) are

$$\begin{aligned}
\{Q, Q^\dagger\} \cdot f &= \nabla_\mu \partial^\mu f = 0 \; , \\
\{Q, Q^\dagger\} \cdot A_\nu \psi^\nu &= \big(\nabla_\nu \nabla_\mu A^\mu + (\pi^2_{\mathbf{7}})_\nu{}^{\gamma\mu\sigma} \nabla_\gamma \nabla_\mu A_\sigma\big)\psi^\nu = 0 \; , \\
\{Q, Q^\dagger\} \cdot \omega_{\mu\nu}\psi^\mu\psi^\nu &= \big((\pi^2_{\mathbf{7}})_{\mu\nu}{}^{\alpha\beta} \nabla_\alpha \nabla^\gamma \omega_{\beta\gamma} + (\pi^3_{\mathbf{1}})_{\mu\nu\rho}{}^{\alpha\beta\gamma} \nabla^\rho \nabla_\alpha \omega_{\beta\gamma}\big)\psi^\mu\psi^\nu = 0 \; .
\end{aligned} \tag{3.16}$$

We have used $\pi^2_7(\omega) = \omega$ to simplify the last expression above.

## 4. Open string moduli

In a general topological theory one can use elements of degree one cohomology to deform the theory using descendant operators. If $\mathcal{O}$ is a degree one operator, in the A/B-model this means that it has ghost number one, whereas in the $G_2$ string this means that it corresponds to one '+' conformal block. Then one can deform the action by adding a term $\int_{\partial\Sigma}\{G^\uparrow_{-\frac{1}{2}}, \mathcal{O}\}$, which is $Q = G^\downarrow_{-\frac{1}{2}}$ closed and of degree 0. Thus the elements of $H^1_Q$ cohomology should correspond to possible deformations of the theory or tangent vectors to the moduli space of open topological $G_2$ strings.

Since open strings correspond to supersymmetric[6] branes, the full moduli space should include both the moduli space of the field theory on the brane as well as the geometric moduli of the branes. For $G_2$ manifolds the latter are simply the moduli of associative and coassociative 3- and 4-cycles, respectively, which have been studied in [18]. Below

[6] In the sense of preserving the extended worldsheet superalgebra.

we will show that the operators $\mathcal{O}$ corresponding to normal modes do satisfy the correct constraints to be deformations of the relevant calibrated submanifolds. Since a priori it is not known what the field theory on these branes will be, in the topological case we will study the constraints on the tangential modes (which in physical strings would correspond to gauge fields on the brane), and attempt to interpret these as infinitesimal deformations in the moduli space of some gauge theory on the brane.

### 4.1 Calibrated geometry

In order to preserve the extended symmetry algebra (such as $\mathcal{N}=2$ or $G_2$) of the worldsheet SCFT in the presence of a boundary, certain constraints must be imposed on the worldsheet currents. These have been studied in [24, 15], and more extensively in [25, 26, 16]. One imposes the boundary condition on the left- and right-moving components of the worldsheet fermions, $\psi^\mu_L = R^\mu_\nu(X)\psi^\nu_R$, and then conservation of the worldsheet currents in the presence of the boundary implies that, on the subspace $M$ where open strings can end,

$$\begin{aligned} \phi_{\mu\nu\sigma} &= \eta_\phi R^\alpha_\mu R^\beta_\nu R^\gamma_\sigma \phi_{\alpha\beta\gamma}\ , \\ (*\phi)_{\mu\nu\sigma\lambda} &= \eta_\phi R^\alpha_\mu R^\beta_\nu R^\gamma_\sigma R^\rho_\lambda (*\phi)_{\alpha\beta\gamma\lambda} \det(R) \\ &= R^\alpha_\mu R^\beta_\nu R^\gamma_\sigma R^\rho_\lambda (*\phi)_{\alpha\beta\gamma\lambda}\ . \end{aligned} \tag{4.1}$$

Note that $R^\alpha_\mu(X)$ (for any $X\in M$) is generally a position-dependent invertible matrix, but locally it can be diagonalized with eigenvalues $+1$ in Neumann directions and $-1$ in Dirichlet directions. $\eta_\phi = \pm 1$ gives two different possible boundary conditions with the choice of $\eta_\phi = 1$ corresponding to open strings ending on a calibrated 3-cycle, while $\eta_\phi = -1$ corresponds to strings on a calibrated 4-cycle [15]. Calibrated submanifolds, first studied in [27], are characterized by the property that their volume form induced by the metric in the ambient space is the pull-back of particular global forms, in this case $\phi$ (for associative 3-cycles) or $*\phi$ (for coassociative 4-cycles). This implies the volume of the calibrated submanifold is minimal in its homology class.

**Remark.** There are several subtleties regarding boundary conditions in topological sigma-models that deserve to be mentioned. Below, we will advocate the perspective that any boundary condition preserving the extended algebra[7] should also be a boundary condition of the topological theory, because the presence of an extended algebra allows one to define a twisted theory. In the A- and B-model, however, although both the A- and B-brane boundary conditions preserve the $\mathcal{N}=2$ algebra, each is compatible with only one of the twists, so a given topological twist is not necessarily compatible with an arbitrary algebra-preserving boundary condition. Moreover, a given topological twist might only depend on the existence of a subalgebra of the full extended algebra, so may be possible even with boundary conditions that do not preserve the full extended algebra. A concrete example of this is the Lagrangian boundary condition for the A-model branes proposed by Witten [13]. This condition is considerably less restrictive that the special Lagrangian

[7] To be precise the boundary conditions preserve some linear combination of the extended algebra in the left/right sector of the worldsheet. So a brane may reduce an $\mathcal{N}=(2,2)$ theory to an $\mathcal{N}=2$ theory.

condition required to preserve the full $\mathcal{N}=2$ algebra in the physical string [24] and reflects the fact that the A-model is well-defined for any Kähler manifold and does not require a strict Calabi-Yau target space. While similar subtleties might, in principle, exist for the $G_2$ twist they are concealed by the fact that the twist does not have a classical realization that we know of. So we will tentatively assume the correct boundary conditions are those that preserve the full $G_2$ algebra on one half of the worldsheet theory.

### 4.2 Normal modes

Let us now consider the cohomology of open strings ending on a D-brane which wraps either an associative 3-cycle or a co-associative 4-cycle. We adopt the convention that $I, J, K, \ldots$ are indices normal to the brane while $a, b, c, \ldots$ are tangential, and Greek letters run over all indices. The state $A_\mu \psi^\mu$ decomposes into normal and tangential modes which will be denoted $\theta_I \psi^I$ and $A_a \psi^a$ respectively; all momenta is tangential, denoted by $k_a$. The normal modes will have the form $\mathcal{A} = \theta_I(X^a)\psi^I$ so $G_{-1/2} \cdot \mathcal{A} = \partial_a \theta_I(X^b)\psi^a \psi^I$. Here $\mathcal{A}$ will denote a *general* operator/state in the CFT and should not be confused with the gauge field (or operator) $A_\mu \psi^\mu$.

**Associative 3-cycles.** Let us now consider the $Q$-cohomology when restricted to an associative 3-cycle $M$. On the 3-cycle the form $\phi$ must satisfy [16]

$$\phi_{\mu\nu\sigma} = R^\alpha_\mu R^\beta_\nu R^\gamma_\sigma \phi_{\alpha\beta\gamma} \ . \tag{4.2}$$

Since $M$ is associative, $\phi$ acts as a volume form on this cycle and, from the above, it is only non-vanishing for an odd number of tangential indices[8]

$$\begin{aligned} \phi_{abc} &= \epsilon_{abc} \ , \\ \phi_{Ibc} &= 0 \ , \\ \phi_{IJK} &= 0 \ . \end{aligned} \tag{4.3}$$

The $Q$-closure of normal modes is given by (3.5)

$$\phi_{bK}{}^J \phi_J{}^{aI} \nabla_a \theta_I = 0 \ , \tag{4.4}$$

where the index structure is enforced by the requirement that $\phi$ has an even number of normal indices.

To understand the geometric significance of equation (4.4) in the abelian theory, recall that $\theta^I$ is just a section of the normal bundle $NM$ of $M$ in $Y$, which by the tubular neighborhood theorem can be identified with an infinitesimal deformation of $M$. This equation is the linear condition on $\theta^I$ such that the exponential map (defined by flowing along a geodesic in $Y$ defined by $\theta^I$) $\exp_\theta(M)$ takes $M$ to a new associative submanifold $M'$. This is just a reformulation of the condition given in [18].

[8] Here, and throughout the paper, we will take $\epsilon$ to be the volume form on the (sub)manifold not merely the antisymmetric tensor.

In [18] McLean defines a functional on the space of (integrable) normal bundle sections by

$$F_\gamma(\theta) = (*\phi(x))_{\mu\nu\rho\gamma} \frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}\frac{\partial x^\rho}{\partial\sigma^c}\epsilon^{abc} \propto (*\phi(x))_{\mu\nu\rho\gamma} \frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}\frac{\partial x^\rho}{\partial\sigma^c}\phi^{abc} \ . \tag{4.5}$$

Here $x(t,\theta,\sigma) = \exp_\theta(\sigma, t)$ is a geodesic curve parameterized by the variable $0 < t < t_1$, which starts at a point $\sigma \in M$ with $\dot{x}(\sigma) = \theta$ at $t = 0$, and flows after a fixed time to $x(t = t_1, \theta, \sigma) \in M'$, the new putative associative submanifold. The functional is just the pull-back[9] of $*\phi$ from $M'$ to $M$ and it should vanish if $M'$ is associative.

For $M'$ to be a associative it turns out to be sufficient to require that the time derivative of $F$ at $t = 0$ vanishes, which gives

$$\dot{F}_\gamma(\theta)|_{t=0} = (*\phi(x))_{Ibc\gamma}\partial_a\theta^I\phi^{abc} = \phi_{I_\gamma}{}^a\partial_a\theta^I. \tag{4.6}$$

This is equivalent to (4.4) since each choice of $bK$ indices in that equation gives only one non-vanishing term. The space of such deformations is generally not a smooth manifold and currently the moduli space of associative submanifolds of a given $G_2$ manifold is not well understood (but see [28] for some recent work on this).

At first glance (4.4) looks like the linearized equation (4.7) in [5] but the fields in that action are actually embedding maps which are non-linear, whereas the $\theta^I$ above are more closely related to linearized fluctuations around fixed embedding maps.[10]

**Remark.** The harmonic condition as follows from (3.16) for normal modes is

$$(\pi^2_{\mathbf{7}})^a{}_I{}^{bJ}\nabla_a\nabla_b\theta_J = 0 \ . \tag{4.7}$$

This also has a nice geometrical interpretation as vector fields $\theta^I$ extremizing the action

$$\int_M \langle Q\cdot\theta, Q\cdot\theta\rangle \ , \tag{4.8}$$

on the associative 3-cycle. Theorem 5-3 in [18] shows that the zeros of this action (which are extrema since it is positive semi-definite) correspond to a family of deformations through minimal submanifolds.

**Coassociative 4-cycles.** The consideration of the 4-cycle $M$ is similar to that of the 3-cycle, but now in the boundary condition we have $\eta_\phi = -1$, so the non-vanishing components of $\phi$ must have an odd number of normal indices and

$$\phi_{abc} = 0 \ . \tag{4.9}$$

[9] More precisely we are pulling back $\chi \in \Omega^3(Y, TY)$, a tangent bundle valued 3-form, defined using the $G_2$ metric $\chi^\alpha{}_{\mu\nu\rho} = g^{\alpha\beta}(*\phi)_{\beta\mu\nu\rho}$.

[10] In [5], maps $x : \Sigma_3 \to Y$ from an arbitrary three-manifold to a $G_2$ manifold are considered and a functional which localizes on associative embeddings is defined. There a reference associative embedding $x_0$ is chosen and used to define a local coordinate splitting of $x^\mu$ into tangential $x^a$ and normal $y^I$ parts. This is different from the present situation where $\theta^I$ is an infinitesimal normal deformation of an associative cycle. $\theta^I$ can be identified with a section of the normal bundle (via the tubular neighborhood theorem) and is essentially a linear object, whereas the $y^I$ above are a local coordinate representation of a non-linear map. Basically $\theta^I$ here are related to the linear variation $\delta y^I|_{x_0}$ (evaluated at $x = x_0$) in [5].

Let us first consider the $Q$-closure of $\theta_I$

$$\phi_{Ic}{}^{b}\phi_{b}{}^{aJ}\partial_{[a}\theta_{J]} = 0 \ . \tag{4.10}$$

These are 24 equations depending on a choice of $I$ and $c$. Examining the index structure, (4.10) reduces to 4 independent equations

$$\phi_{b}{}^{aJ}\nabla_a\theta_J = 0 \ , \tag{4.11}$$

where we replaced the commutator of a derivative with the covariant derivative on $M$ in the induced metric.

Following [18], let us observe an isomorphism between the normal bundle $NM$ of the 4-cycle $M$, and the space of self-dual 2-forms $\Lambda^2_+(M)$ on $M$, given by

$$\theta^I \to \theta^I\phi_{Iab} \equiv \Omega_{ab} \ , \tag{4.12}$$

$$\Omega_{ab} \to \phi^{Iab}\Omega_{ab} = \phi^{Iab}\phi_{Jab}\theta^J = \frac{1}{6}\theta^I \ , \tag{4.13}$$

where we have used the first identity in (A.3).

To see that $\Omega_{ab}$ is self-dual we use the second identity in (A.3) and the fact that $*\phi_{ab}{}^{cd} \propto \epsilon_{ab}{}^{cd}$ on $M$, so that

$$(*_4\Omega)_{ab} \propto *\phi_{ab}{}^{cd}\Omega_{cd} = \phi_{ab}{}^{cd}\theta^I\phi_{Icd} = \frac{1}{6}\phi_{Iab}\theta^I = \frac{1}{6}\Omega_{ab} \ . \tag{4.14}$$

Let us now use (4.13) to see what (4.11) implies for $\Omega_{ab}$;

$$0 = \phi_{b}{}^{aJ}\nabla_a\phi_J{}^{cd}\Omega_{cd} = \nabla_a\left(\phi_{b}{}^{aJ}\phi_J{}^{cd}\Omega_{cd}\right) = \nabla^a\left[\left(\frac{1}{9}\Omega_{ba} + \frac{1}{18}\Omega_{ba}\right)\right] = \frac{1}{6}\nabla^a\Omega_{ba} \ . \tag{4.15}$$

This equation is just $d^\dagger\Omega = 0$, and since $\Omega$ is self-dual, it also implies $d\Omega = 0$ so that $\Omega$ must be harmonic. Thus the $Q$-cohomology for the normal modes is given by $\theta^I$ which map to harmonic self-dual 2-forms on $M$.

Since the $Q^\dagger$-cohomology on the normal modes is trivial (eq. (3.8) is trivially true for normal directions), such $\theta^I$ are $Q$-closed and co-closed, and hence $Q$-harmonic. Thus their $Q$-cohomology is isomorphic to the de Rham cohomology group $H^2_+(M)$ of harmonic self-dual 2-forms on $M$. This corresponds to the geometric moduli space of deformations of a coassociative 4-cycle, determined by McLean in [18].

### 4.3 Tangential modes

For the tangential modes the $Q$- and $Q^\dagger$-closure conditions are just (3.5) and (3.8) with all the indices replaced by worldvolume indices $a, b, c, \ldots$.

**Associative 3-cycles.** On the 3-cycle it is convenient to represent the $Q$-closure condition using the projector $\pi^2_{\mathbf{7}}$ in terms of $\phi$ which gives

$$\phi_{ab}{}^{c}\phi_{c}{}^{de}\partial_{[d}A_{e]} = 0 \ . \tag{4.16}$$

When pulled back to the associative cycle, $\phi$ is proportional to the volume form and so this is

$$\epsilon_{ab}{}^{c}\epsilon_{c}{}^{de}\partial_{[d}A_{e]} = 0 \ , \tag{4.17}$$

which is just multiple copies of the equation $\partial_{[d}A_{e]} = 0$. Therefore any tangential deformation corresponds to a flat connection on the 3-cycle.

Requiring the deformation $A_a\psi^a$ be also be $Q^\dagger$-closed, and hence a harmonic representative of $Q$-cohomology, implies (3.8), which can be viewed as enforcing a covariant gauge condition.

Combined together this means that the $Q$-cohomology for tangential modes on $M$ is spanned by the space of gauge-inequivalent flat connections on $M$. This matches the result for Lagrangian submanifolds in the A-model and also the results derived using $\kappa$-symmetry for physical branes in [29].

**Coassociative 4-cycles.** On the 4-cycle it is easier to use the representation of $Q$-closure

$$\left(\left(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d\right) + 24(*\phi)^{ab}{}_{cd}\right)\partial_{[a}A_{b]}\psi^c\psi^d = 0 \ , \tag{4.18}$$

in terms of the 4-form $*\phi$, which is now proportional to the volume form on $M$. Defining $F_{ab} = \partial_{[a}A_{b]}$ to be the field strength of the U(1) gauge field, the equation above implies

$$(*_4F)_{ab} = 12(*\phi)^{cd}{}_{ab}F_{cd} = -F_{ab} \ . \tag{4.19}$$

Thus $F_{ab}$ is constrained to be anti-self-dual (ASD) on $M$. Therefore any tangential deformation on the 4-cycle is given by a gauge field with ASD field strength. Note an important difference with the case of normal modes. In the latter case each $\theta^I$ is mapped uniquely to a harmonic self-dual 2-form $\Omega_{ab}$ on $M$, so there are exactly $b^2_+(M)$ such modes. In this case however the tangential mode $A_a$ is the potential for a gauge field with ASD field strength (i.e. an (anti-)instanton configuration). Hence the tangential modes correspond to tangent vectors on the moduli space of instanton configurations on $M$.

Again the condition $\nabla_aA^a = 0$ for $Q^\dagger$-closure is simply a gauge choice, implying that each $Q$-harmonic representative is associated to a unique orbit of the gauge group (up to Gribov ambiguity in the path integral). In fact, these harmonic constraints $*_4F = -F$, $d^\dagger A = 0$ are precisely (linearized versions of) the conditions cited in equation (5.22) of [30] as defining the deformations of an instanton configuration.

In physical string theory the anti-self-duality constraint on the field strength of a coassociative brane has been determined in [29] using $\kappa$-symmetry of the DBI action. In [31], a topological field theory is proposed on calibrated 4-cycles whose total moduli space is a product of the moduli space of geometric deformations with the moduli space of ASD connections on $M$. We will see shortly that this is indeed the worldvolume theory on coassociative 4-cycles for the open $G_2$ string.

## 5. Scattering amplitudes

Before considering the nature of the worldvolume theory of the calibrated 3- and 4-cycles, it will be useful to consider some scattering amplitudes in the open $G_2$ theory, as these can be compared with field theoretic scattering amplitudes and will help constrain the interaction terms in the worldvolume action. In fact, as will be discussed in the next section, these interactions can actually be related to string field theory, not just to effective field theory, if one concedes that the $G_2$ string is independent of its coupling constant, as argued in [8].

### 5.1 3-point amplitude

The simplest amplitudes to calculate (and the only ones we will need) are the 3-point functions of degree one fields $A_\mu \psi^\mu$, which are essentially already calculated in [8]. Introducing Chan-Paton factors into the calculation performed there gives the 3-point function of three ghost number one fields as

$$\lambda^3 \frac{3}{2} f_{jik} \int_Y \phi^{\alpha\beta\gamma}(x) A^i_\alpha(x) A^j_\beta(x) A^k_\gamma(x) , \tag{5.1}$$

where $f_{ijk}$ are the structure functions for the Lie algebra of the gauge group $G$ and $\lambda$ is the normalization of the bulk-boundary 2-point function in the $G_2$ CFT (these are generally not relevant and will not be treated with a great deal of care).

**Tangential modes.** For an associative 3-cycle embedding $i : M \to Y$, we have the relation $i^*(\phi) = \epsilon$, where $\epsilon$ is the volume form on $M$. If we now consider the previous calculation but where now the fields $\psi^\mu$ are restricted to be along the 3-brane (so they have indices $a, b, c \ldots$), we find that

$$\langle AAA \rangle = \lambda^3 \frac{3}{2} f_{jik} \int_M \epsilon^{abc}(x) A^i_a(x) A^j_b(x) A^k_c(x) . \tag{5.2}$$

As will be discussed in the next section, this is an interaction vertex for Chern-Simons theory, which is the part of the effective worldvolume theory for the 3-cycle.

As mentioned in previous section, on a coassociative 4-cycle $\phi^{abc} = 0$ so the 3-point function of tangential modes vanishes.

**Normal and mixed modes.** We can now try to consider a mixture of normal or tangential modes in the 3-point function. The boundary conditions on the open $G_2$ string, preserving the extended algebra on a 3-cycle, imply [15] that only $\phi^{abc}$ and $\phi^{IJc}$ are non-vanishing. Thus $\phi$ is only non-vanishing for an even number of indices in Dirichlet directions, so we can only scatter two normal modes and one tangential mode. This gives

$$\langle \theta\theta A \rangle = \lambda^3 \frac{3}{2} f_{jik} \int_M \phi^{IJc}(x) \theta^i_I(x) \theta^j_J(x) A^k_c(x) . \tag{5.3}$$

On a 4-cycle the non-vanishing components of $\phi$ have an odd number of normal indices, and it is easy to see that the only non-vanishing 3-point functions of degree one modes are

$$\begin{aligned} \langle \theta AA \rangle &= \lambda^3 \frac{3}{2} f_{jik} \int_M \phi^{Iab}(x) \theta^i_I(x) A^j_a(x) A^k_b(x) , \\ \langle \theta\theta\theta \rangle &= \lambda^3 \frac{3}{2} f_{jik} \int_M \phi^{IJK}(x) \theta^i_I(x) \theta^j_J(x) \theta^k_K(x) . \end{aligned} \tag{5.4}$$

## 6. Worldvolume theories

We have already determined the BRST cohomology of normal and tangential modes on 3- and 4-cycles. These should be thought of as marginal deformations of the theory preserving the twisting on the worldsheet (by general arguments that map an element of BRST cohomology to a descendant that can generate a deformation). When considered from the spacetime perspective, the elements of BRST cohomology should translate into spacetime fields and we expect the BRST closure condition to correspond to the *linearized* spacetime equations of motion. This is true in physical string theory and can be derived more rigorously via open string field theory for topological strings, as will be reviewed below.

For the normal modes, the BRST cohomology condition can be translated into constraints on deformations of the calibrated submanifolds, such that these modes correspond to tangent vectors on the moduli space of (co)associative cycles in the $G_2$ manifold.

For tangential modes, the BRST cohomology condition looks different for the different cycles. On the 3-cycle, BRST closure and co-closure of the tangential mode $A_a$ imply $dA = 0$ and $d^\dagger A = 0$, so that $A$ is a flat connection in a fixed gauge, and we expect a gauge theory whose solutions correspond to gauge-inequivalent flat connections. On the 4-cycle, BRST closure and co-closure of $A_a$ imply

$$*_4 dA = -dA \;, \qquad\qquad d^\dagger A = 0 \;. \tag{6.1}$$

These equations are the linearization of the condition for a variation of a gauge field to be a deformation of an instanton solution (c.f. equation (5.50) in [30]). This suggests, in analogy with the geometric moduli, that the theory on the worldvolume should be a gauge theory extremizing on instantons and that marginal tangential deformations of the worldsheet theory should correspond to tangent vectors on the moduli space of instantons.

In the case of both the 3- and 4-cycle, the worldvolume theory will include contributions from the normal and tangential modes, and so should result in a theory whose moduli space includes the normal and tangential deformations that we have determined in section 4. We also expect that the other physical states, which are massless in the twisted theory, may still play a role in the spacetime action even though they cannot be used to generate boundary deformations of the CFT,[11] and hence are not moduli of the theory.

To determine the relevant spacetime actions and how the normal and tangential moduli, as well as the higher ghost number fields, come into play we will start by considering Witten's derivation of Chern-Simons theory from open string field theory (OSFT). We will find that by restricting our attention to tangential modes on a calibrated 3-cycle we can re-derive Witten's Chern-Simons theory simply by following the arguments of [13]. We will then attempt to generalize this derivation to include normal modes. Their contribution is expected to be related to the topological theories in [5, 6], whose actions also localize on the moduli space of associative 3-cycles (though, as we will see, this relation is mostly at the level of equations of motion). Following a comment in [13], we expect the higher string

[11] Only a ghost number one state has a 1-form descendant with ghost number 0; ghost number $p$ states have $p$-form descendants with ghost number zero, so to preserve the ghost number in the worldsheet action we would have to integrate them over a $p$-cycle on the worldsheet.

modes to be related to additional fields generated by gauge-fixing the CS action. This is discussed in appendix B.

Once we have transplanted Witten's arguments for special Lagrangian branes in a Calabi-Yau to associative branes in a $G_2$ manifold, we will apply them to branes wrapping coassociative cycles and branes wrapping all of $Y$.

### 6.1 Chern-Simons theory as a string theory

In [13] Witten argues that the open A-model on $T^*M$ reduces exactly to Chern-Simons theory on $M$, for any 3-manifold $M$. There are several arguments supporting this claim and we will attempt to generalize them below to the $G_2$ case. Before doing so, we first review them briefly.

The first argument concerns $Q$-invariance of a boundary term in the string path integral. In general the open string path integral can be augmented by coupling to a 'classical' background gauge field. This is done by including an additional piece in the integrand of the path integral which is of the form

$$\mathrm{Tr} P \exp \left( \oint_{\partial\Sigma} X^*(A) \right) . \tag{6.2}$$

Here $A$ is a (non-abelian) connection defined on the brane $M$ and the term above is a Wilson loop for the pull-back of this connection along the boundary of the worldsheet $\Sigma$. Requiring that this new term preserve the $Q$-invariance of the action implies that the field strength $F = dA + A \wedge A$ must vanish. Hence open strings in the A-model can only couple to flat connections.

To more rigorously establish that the relevant spacetime theory is Chern-Simons theory, Witten considers the OSFT action

$$\int \mathcal{A} \star Q\mathcal{A} + \frac{2}{3} \mathcal{A} \star \mathcal{A} \star \mathcal{A} , \tag{6.3}$$

where $\mathcal{A}$ is a functional of the open string modes quantized on a fixed time slice, and $Q$ is the appropriate BRST operator of the theory. The integration measure is defined by the path integral over the disc.[12] The linearized equations of motion (coming from the quadratic part of the OSFT action) enforce the requirement that physical states are BRST-closed on-shell:

$$Q\mathcal{A} = 0 . \tag{6.4}$$

In the large coupling constant limit ($t \to \infty$) the $Q$-cohomology can be studied by restricting to functionals $\mathcal{A}$ that depend only on the string zero-modes, $X_0^\mu$ and $\psi_0^\mu$. The BRST operator, $Q$, acting on such states, reduces to the exterior derivative $d$ on $T^*M$ (which we can write in terms of the zero modes)

$$d = dx^\mu \frac{\partial}{\partial x^\mu} = \psi_0^\mu \frac{\partial}{\partial X_0^\mu} . \tag{6.5}$$

[12] There is a subtlety here. In OSFT for the bosonic string this measure involves gluing together several discs using conformal transformations, but in the setting of a topological theory all the states have conformal weight zero under the twisted stress-tensor so they do not transform under conformal transformations.

Since the $t \to \infty$ limit is exact in the A-model (modulo world-sheet instantons which are not present when the target space is $T^*M$), these identifications are not approximations but rather exact statements. This allows one to identify the string field action with Chern-Simons theory.

To make this identification one must identify the string field $\mathcal{A}$ with the target space gauge field $A_\mu(x)dx^\mu$. The general form for $\mathcal{A}$ at large $t$ is given by the expansion

$$\mathcal{A}(X^\mu, \psi^\mu) = f(X_0) + A_\mu(X_0)\psi_0^\mu + \beta_{\mu\nu}(X_0)\psi_0^\mu\psi_0^\nu + C_{\mu\nu\rho}(X_0)\psi_0^\mu\psi_0^\nu\psi_0^\rho \,, \tag{6.6}$$

in 3 dimensions. The reason that $\mathcal{A}$ reduces to $A_\mu(X_0)\psi_0^\mu$ is simply that only ghost number one string fields should be considered, and here ghost number coincides with fermion number. Witten comments that it is possible to relate the other terms in the expansion to ghost and anti-ghosts fields derived from gauge-fixing CS theory [32], or alternatively gauge-fixing OSFT. In appendix B we will show that this is indeed the case when we repeat this derivation on an associative cycle in a $G_2$ manifold.

Witten provides a final argument for CS theory as the string field theory for the A-model, namely that the open string propagator on the strip reduces to the CS propagator in the large $t$ limit. This is essentially the statement that $\frac{b_0}{L_0} = \frac{d^\dagger}{\Box}$. For the topological string, $b_0$ is replaced by the superpartner of the stress-energy tensor in the twisted theory (i.e. $Q^\dagger$ in $T = \{Q, Q^\dagger\}$). In the $G_2$ case this would be (tentatively) $G^\uparrow_{-\frac{1}{2}}$ [8].

We will now attempt to establish the validity of these arguments for the open $G_2$ string ending on a calibrated 3-cycle. Before doing so we should mention that what was missing in this treatment is a discussion of the normal modes on the brane. It is not immediately clear whether these modes modify the Chern-Simons action on the special Lagrangian cycle (though one would imagine they should in order to capture the dependence of the theory on the geometric moduli of the brane).

### 6.2 Chern-Simons theory on calibrated submanifolds

If we consider only the tangential modes on a calibrated cycle then the $Q$-closure conditions become (in the free field approximation)

$$\partial_a f(X) = 0 \,, \tag{6.7}$$

$$\epsilon^{abc}\partial_a A_b = 0 \,,$$

$$\epsilon^{abc}\partial_a \beta_{bc} = 0 \,, \tag{6.8}$$

for the degree 0, 1, and 2 components of the string field. Here we have already used that $\phi_{abc} \propto \epsilon_{abc}$ on the 3-cycle. This is consistent with the notion that $Q = G^\downarrow_{-\frac{1}{2}} = d$ in the large $t$ limit. More generally, the complex (2.5), which encodes the BRST cohomology, reduces, when restricted to the tangential directions on an associative 3-cycle, to the de Rham complex so $Q = d$ and $Q^\dagger = d^\dagger$.

Recall, from the discussion in section 2.3, that, in contrast to the situation in the A-model, we do not have an explicit worldsheet action to work with and hence do not have a Hamiltonian formulation which might directly establish the $t$ invariance of the action.

Assuming this invariance none-the-less, the equations above imply that the quadratic part of the string field action reduces to the quadratic part of Chern-Simons theory. That is, in the large $t$ limit, the $Q$-closure constraint becomes the linearized CS equation of motion. Here we have also considered modes with fermion number different from one; these will be discussed in appendix B in relation to gauge-fixing Chern-Simons theory.

Also in this limit (of free string theory approximation), the $Q^\dagger$-closure constraints become

$$\begin{aligned} \nabla_a A^a &= 0 \,, \\ \nabla_a \beta^{ab} &= 0 \,. \end{aligned} \tag{6.9}$$

The first term is just the gauge-choice $d^\dagger A = 0$. We will discuss the spacetime interpretation of $\beta_{ab}$ in appendix B and it will be clear why it satisfies this constraint. Let us now translate the rest of Witten's arguments to the $G_2$ case.

The argument is essentially that open string field theory with the action (6.3) reduces to Chern-Simons theory in the large $t$ limit, if one restricts the string field to have ghost number 1 (which, in the $G_2$ case, translates into fermion number 1 because the ghost number is the grading for the $Q$-cohomology, and that is given by the fermion number). That this holds for the kinetic term follows because we have shown that the linearized CS action is the same as the linearized $Q$-closure condition.

For the interaction term this just follows from the fact that the 3-pt function of the ghost number one parts of $\mathcal{A}$ reduces to the wedge products of the Lie algebra valued 1-forms, $A_a(x)dx^a$. This is because, at large $t$, $\mathcal{A}$ depends only on the zero modes so the ghost number one part has the form $A_a(X_0)\psi_0^a$ which can be mapped to one-forms in spacetime. We show in section 5.1 that the 3-pt function of these modes is just the 3-pt correlator of CS theory.

Witten also shows that the propagator of the OSFT reduces, in the $t \to \infty$ limit, to the CS propagator. We will reproduce this argument briefly here for the $G_2$ case. A much more complete treatment (of the analogous A/B-model argument) can be found in section 4.2 of [13]. The open string propagator is simply given by the partition function of a finite strip, of length $T$ and width 1 with the standard metric

$$ds^2 = d\sigma^2 + d\tau^2 \,. \tag{6.10}$$

In OSFT the moduli space of open Riemann surfaces is built by gluing such strips together. The strip has one modulus, namely its length, so in calculating the partition function, one insertion of $G^\uparrow_{-\frac{1}{2}}$ folded against a Beltrami differential $\mu$ is required [8]

$$\int d\sigma d\tau \; \mu(\sigma,\tau) G^\uparrow(\sigma,\tau) \,. \tag{6.11}$$

The Beltrami differential here is just given by a change to the metric that changes the length of the strip and is given by a function $f(\tau) = \delta T \cdot \delta(\tau - \tau_0)$ for any $\tau_0$ on the strip. Here $\delta T$ is the infinitesimal change in the length of the strip generated by this differential.

Thus the insertion becomes

$$\int d\sigma d\tau\ \delta T \cdot \delta(\tau - \tau_0) G^{\uparrow}(\sigma, \tau) = \delta T \int d\sigma\ G^{\uparrow}(\sigma, \tau_0)\ . \tag{6.12}$$

Because we have been working in the NS sector, the integral of the current $G^{\uparrow}(z)$ around a contour (given by fixed $\tau_0$ which maps to a half-circle in the complex plane) will just give a $G^{\uparrow}_{-\frac{1}{2}}$ insertion in the world-sheet path integral, so its overall form is

$$\int_0^{\infty} DT \left( G^{\uparrow}_{-\frac{1}{2}} \right) e^{-TL_0} = \frac{G^{\uparrow}_{-\frac{1}{2}}}{L_0}\ . \tag{6.13}$$

By our previous identification of $G^{\uparrow}_{-\frac{1}{2}}$ with $d^{\dagger}$ (this becomes $d^{\dagger}$ on $M$ for tangential modes) and $L_0$ with $\Box$ in the large $t$ limit, this becomes $\frac{d^{\dagger}}{\Box}$ which is the CS propagator [13]. One should note that in the A-model this follows rather directly but in the $G_2$ string it depends on the fact that $\phi_{abc} \propto \epsilon_{abc}$ on the associative cycle (so, as previously mentioned, $Q = \check{D}$ reduces to $d$) and thus, in particular, might not hold on a coassociative cycle.

There is a final argument one can make in favour of CS theory, though it is more heuristic. We want to argue, as Witten has, that coupling the worldsheet to a classical background gauge field via a term such as (6.2) requires this background to satisfy $F = 0$ which is the equation of motion for Chern-Simons theory.

In [8], a heuristic version of the twisted $G_2$ action is derived using the decomposition of worldsheet fermions into $\uparrow$ and $\downarrow$ components, $\psi = \psi^{\uparrow} + \psi^{\downarrow}$. This is heuristic because this decomposition is essentially quantum and is not understood at the level of classical fields. Using this decomposition we can check Witten's argument for the BRST-invariance of a boundary coupling to a classical configuration of the gauge field

$$\mathrm{Tr} P \exp \left( \oint_{\partial\Sigma} A_\mu \partial_t X^\mu \right)\ . \tag{6.14}$$

The variation of this factor in the partition function under $[Q, X^\mu] = \delta X^\mu$ is given by

$$\mathrm{Tr} P \oint_{\partial\Sigma} \delta X^\mu \partial_t X^\nu F_{\mu\nu} d\tau \cdot \exp \left( \int_{\partial\Sigma;\tau} A_\mu \partial_t X^\mu \right)\ , \tag{6.15}$$

where the contour in the exponent must start and end at the point $\tau$ [13]. To make this variation vanish requires that the first term vanish and since [8]

$$\delta X^\mu = i\epsilon_L \psi_L^{\downarrow\mu} + i\epsilon_R \psi_R^{\uparrow\mu}\ , \tag{6.16}$$

this implies that $F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu] = 0$ for classical configurations of the background gauge field $A$. This is, of course, the Chern-Simons equation of motion.

In the physical theory one could also couple to a term of the form $C_{\mu\nu}\psi^\mu\psi^\nu$, but no such terms seem to effect the derivation of $F = 0$ above in the A-model, because any such coupling results in a variation which cannot cancel the gauge-field coupling.

The only boundary term in a topological theory should be generated by the descent procedure starting from a $Q$-closed ghost number one field whose descendent is a ghost number zero one-form that is given by

$$\{G^{\uparrow}_{-\frac{1}{2}}, A_\mu \psi^\mu\} = A_\mu \partial_t X^\mu + \pi^2_{\mathbf{14}}(\partial_\mu A_\nu \psi^\mu \psi^\nu) \ . \tag{6.17}$$

Both these terms have conformal weight 1 and, by virtue of a standard descent argument, are $Q$-closed up to a total derivative. To apply Witten's argument here it is necessary to understand why the second term cannot appear on the boundary. This follows because we are considering modes tangential to an associative cycle and one can check that on such a cycle $\Lambda^2 T^* M \subset \iota^*(\Lambda^2_7(Y))$ (here $\iota : M \to Y$ is the embedding of the three cycle into the ambient $G_2$).

To derive the Chern-Simons action we have considered only the ghost number one part of the string field $\mathcal{A}$ as this is the standard prescription in OSFT. In some cases, however, it is desirable to consider the full expansion of $\mathcal{A}$ and include fields of all ghost number in the action. This is because the higher modes just play the role of ghosts in gauge-fixing the OSFT action [33]. This is a special feature of Chern-Simons like theories [32] and so will apply for all the brane theories that we derive. We include an appendix B describing the general form of the gauge-fixed actions for these theories that we will need when we consider their one-loop partition functions.

### 6.3 Normal mode contributions

In the previous section we argued that the tangential modes of the $G_2$ worldsheet correspond to gauge fields in a CS theory on the 3-cycle and when higher string modes are included this becomes gauge-fixed CS theory.

We are also interested in terms in the effective action that include the normal modes. The most direct way to to get at a normal mode action is to simply expand the terms $\mathcal{A} \star Q\mathcal{A}$ and $\mathcal{A} \star \mathcal{A} \star \mathcal{A}$ in the OSFT action. Ignoring the higher string modes, we have

$$\begin{aligned} \mathcal{A} &= A_a \psi^a + \theta_J \psi^J \ , \\ Q\mathcal{A} &= \{Q, A_a \psi^a\} + \{Q, \theta_I \psi^I\} \\ &= \phi_{IJc}\phi^{cde}\nabla_d A_e \psi^I \psi^J + \phi_{abc}\phi^{cde}\nabla_d A_e \psi^a \psi^b + \phi_{aIJ}\phi^{JbK}\nabla_b \theta_K \psi^a \psi^I \ . \end{aligned} \tag{6.18}$$

Recall that the integration of expressions involving string fields, $\mathcal{A}$, in the OSFT action corresponds to evaluating the correlator of the integrand, decomposed in individual string modes on the disc. In the $G_2$ string only certain combinations of string modes will have a non-vanishing 3-pt function depending on the conformal blocks the modes correspond to (see [8]). In our calculation of the 3-pt functions above, this translates into non-vanishing 3-pt functions when we can contract the spacetime indices of the string modes with the 3-form $\phi$. From our previous calculation of three point functions in sections 5.1 (see also appendix B.1.2) we find the generic form of a 3-pt function on the disc

$$\begin{aligned} \langle \lambda\omega \rangle &= \int_M \phi^{\mu\nu\rho} \mathrm{Tr}\left(\lambda_\mu \omega_{\nu\rho}\right) \ , \\ \langle \alpha\beta\gamma \rangle &= \int_M \phi^{\mu\nu\rho} \mathrm{Tr}\left(\alpha_\mu \beta_\nu \gamma_\rho\right) \ , \end{aligned} \tag{6.19}$$

(where, e.g. $\omega = 1/2\,\omega_{\mu\nu}(x)\psi^\mu\psi^\nu$). Doing this gives the following action

$$S_{\text{deg }1} = \int_M \phi^{abc}\,\mathrm{Tr}\left(A_a\nabla_b A_c + \frac{2}{3}A_aA_bA_c\right) + \phi^{IaJ}\,\mathrm{Tr}\big(\theta_I(\nabla_a\theta_J + [A_a,\theta_J])\big)\ , \tag{6.20}$$

where the trace Tr is over Lie algebra indices. The interaction terms can be calculated directly in string perturbation theory by checking 3-pt disc amplitudes whereas the kinetic terms coming from $\mathcal{A}\star Q\mathcal{A}$ vanish in perturbation theory because on-shell string modes satisfy $Q\mathcal{A}=0$. To determine them we either simply consider all the terms of the correct degree in the string mode decomposition of $\mathcal{A}\star Q\mathcal{A}$ or 'formally' calculate 3-pt functions assuming the field $\mathcal{A}$ is off-shell. Both result in the same action and as a consistency check, the linearized equations of motion for this action correspond to the BRST closure of the string modes. We have not been too careful with the coefficients in (6.20) but this is because most coefficients either follow from gauge invariance or can be absorbed into field redefinitions.

The equations of motion for this action are

$$\epsilon^{abc}F_{bc} = \phi^{IaJ}[\theta_I,\theta_J]\ , \tag{6.21}$$

$$\phi^{aIJ}\big(\nabla_a\theta_J + [A_a,\theta_J]\big) = 0\ . \tag{6.22}$$

In the abelian case this just reduces to $F=0$ and the geometric constraint (4.11) on the normal modes describing associative deformations. In the non-abelian case this is no longer true but of course in this setting we have lost the simple association of $\theta_I$ with normal deformations of the brane, as the string modes become matrix-valued.

At first glance the equations above look similar in form to the Seiberg-Witten type equations (32) and (40) in [28]. This reference is concerned with resolving the singular structure of the moduli space of deformations of associative submanifolds in a general $G_2$ manifold by considering a larger space of deformations where one is allowed to also deform the induced connection on the normal bundle to make the deformed submanifold associative. This amounts to a choice of complex structure on the normal bundle, for each deformation of the 3-submanifold, such that its reduced structure group $\mathrm{U}(2)\subset\mathrm{SO}(4)$ in the $G_2$ manifold is compatible with the induced metric connection. This additional topological restriction on the $G_2$ manifold is something we have not assumed and indeed, for general gauge group, there is no obvious relation between (6.21), (6.22) and the purely geometric equations in [28].[13]

### 6.4 Anti-self-dual connections on coassociative submanifolds

We expect that the worldvolume theory on the 4-cycle should have equations of motion corresponding to the BRST closure of the associated string modes. Let us consider the following action

$$S[A,\theta] = \int_M \phi^{Iab}\mathrm{Tr}\big(\theta_I F_{ab}\big) + \frac{2}{3}\,\phi^{IJK}\mathrm{Tr}\big(\theta_I\theta_J\theta_K\big)\ . \tag{6.23}$$

[13] It is possible that the $\mathrm{U}(1)$ part of our gauge connection could be related to the $\mathrm{U}(1)\subset\mathrm{U}(2)$ part of the induced connection on the normal bundle with fixed complex structure in [28].

As with the action on a 3-cycle we cannot directly check the quadratic terms by considering a string correlator because the relevant correlators vanish for on-shell states as dictated by the fact that the quadratic terms in the action determine the BRST closure condition. Rather, we can compare the linearized equations of motion (generated purely by the quadratic terms) and the string BRST closure condition and these should match.

The abelian $\theta_I$ equation of motion is now just $\phi^{Iab}F_{ab} = 0$, which implies anti-self-duality of $F$ and so matches the BRST closure condition. The $A_a$ equation of motion is

$$\phi^{abI} D_b \theta_I = 0 \,, \tag{6.24}$$

where $D_a = \nabla_a + [A_a, ]$ on $M$. This equation is more conveniently expressed in terms of the self-dual 2-form $\omega_{ab} = \phi_{abI}\theta^I$ on $M$. At the linear level, the equation above implies $\omega$ is co-closed, and hence also closed since it is self-dual. Thus we have the correct linearized condition for coassociative deformations found by McLean.

We can also consider the formal structure of the term $\mathcal{A} \cdot Q\mathcal{A}$ in the OSFT action, letting $\mathcal{A}$ go 'off-shell', and indeed we find matching.

As a further check we should compare the interaction term to string scattering amplitudes. The 3-pt function for a general degree one vertex operator in the topological theory is given by

$$\lambda^3 \frac{3}{2} \int_Y \phi^{\alpha\beta\gamma}(x) \mathrm{Tr}\big(A_\alpha(x) A_\beta(x) A_\gamma(x)\big) \,. \tag{6.25}$$

On the 4-cycle the only non-vanishing components of $\phi$ must have an even number of tangential indices, which implies the following non-vanishing amplitudes

$$\begin{aligned} &\lambda^3 \frac{3}{2} \int_M \phi^{Iab} \mathrm{Tr}\big(\theta_I A_a A_b\big) \,, \\ &\lambda^3 \frac{3}{2} \int_M \phi^{IJK} \mathrm{Tr}\big(\theta_I \theta_J \theta_K\big) \,. \end{aligned} \tag{6.26}$$

The first line above corresponds to the cubic interaction $\theta AA$ in the first term of (6.23) while second correlator in (6.26) implies the cubic vertex in the second term. This last term, of course, only corrects the non-abelian instanton equation of motion

$$\phi^{Iab} F_{ab} = -\phi^{IJK}[\theta_J, \theta_K] \,, \tag{6.27}$$

and so has no effect on the geometric interpretation in the abelian case.

In [31] Leung proposes a 1-form on the space $\mathcal{C} = \mathrm{Map}(M, Y) \times \mathcal{A}(M)$ where $M$ is a 4-manifold, $Y$ is a $G_2$ 7-manifold and $\mathcal{A}(M)$ is the space of Hermitian connections on the gauge bundle $E \to M$ (with fibre $G$)

$$S(f, D_E)(v, B) = \int_M \mathrm{Tr}\left(f^*(\iota_v \phi) \wedge F_E + f^*(\phi) \wedge B\right) \,. \tag{6.28}$$

Here $(f, D_E) \in \mathcal{C}$ and $(v, B) \in T_{(f,D_E)}\mathcal{C}$ with $v$ a section of $TY$, $B \in \Lambda^1(M, \mathrm{ad}\, G)$ and $F_E$ the curvature of $D_E$ (here $f$ is an element of $\mathrm{Map}(M, Y)$ and should not be confused with the $f$ used to denote the zero fermion component of the string field). The one-form $S$ is invariant under diffeomorphisms of $M$ and its zeros correspond to coassociative embeddings $f(M) \subset Y$ with anti-self-dual connections on them. This follows from the fact that $S$ must vanish when evaluated on arbitrary vectors, $B$, implying $f^*(\phi) = 0$, and arbitrary $v$ implying that $F_E = -{*F_E}$.

To compare with our theory we do not want to consider the space of all such maps, but only the local deformations of a given coassociative $f(M)$ in $Y$, so we only consider fluctuations around a fixed coassociative submanifold. Thus we will take $f$ to be a coassociative embedding implying that the second term in the action above vanishes and $*\phi$ defines the volume form on the embedded coassociative 4-cycle $f(M)$. Thus, we rewrite Leung's functional to generate the following action functional[14]

$$S^0[A,\theta] = \int_M \mathrm{Tr}\left(f^*(\iota_\theta \phi) \wedge F\right) = \int_M \phi^{Iab}\mathrm{Tr}\big(\theta_I\left(\partial_a A_b + A_a A_b\right)\big) \ , \tag{6.29}$$

using the identity $\epsilon_{abcd}\,\phi^{cdI} = 2\,\phi_{ab}{}^I$ on the coassociative cycle.

Thus we see that the open $G_2$ string has reproduced the action Leung suggested in order to study SYZ in the $G_2$ setting and it has also introduced an additional term that is not present in Leung's action.

### 6.5 Seven-cycle worldvolume theory

As in physical string theory, it is natural to expect the 3- and 4-cycle theory to look like the dimensional reduction of a theory on the whole 7-manifold (which is trivially calibrated by its volume form $\phi \wedge *\phi$). Lee et al [34], who propose theories closely related to our 3- and 4-cycle theories, claim that this theory should be related to (deformed) Donaldson-Thomas theory [19].

The 7-cycle theory can be determined exactly the same way as the 3- and 4- cycle theory. For the interaction term we just calculate the 3-pt functions of the (ghost number one) terms in $\langle \mathcal{A} \star \mathcal{A} \star \mathcal{A}\rangle$ given by (6.19). The kinetic terms, defining the linearized equations of motion, should correspond to $Q\mathcal{A} = 0$ and they should match $\mathcal{A} \star Q\mathcal{A}$.

This gives the following action

$$S = \int_Y \phi^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu \partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) = \int_Y *\phi \wedge CS_3(A) \ . \tag{6.30}$$

The equation of motion for this action is

$$*\phi \wedge F = 0 \ . \tag{6.31}$$

This is one of the equations in [19] where it is argued to be associated with the 7-dimensional generalization of Chern-Simons theory. In the abelian theory this equation of motion is simply $\check{D}A = 0$ which has no global solutions which are not exact (i.e. $A = df$) because $H^1_7(Y) = 0$ for $G_2$ manifolds. Of course, as a gauge field $A$ need not be a global one form and then this result no longer applies. This is similar to the situation one finds for Chern-Simons theory on a simply connected manifold.

[14] More precisely, Leung's one-form, $\Phi_0$, descends to a closed one-form on the space '$\mathcal{C}$/Diffeo(M)' and this form is locally the derivative of a functional, $\mathcal{F}$, whose critical points are zeros of $\Phi_0$. Our action is most closely related to this functional.

Note that for the action (6.30) to be gauge invariant under large gauge transformations $*\phi$ must actually be an integral cohomology class. A similar issue arises in holomorphic Chern-Simons theory as mentioned by Nekrasov in [2] but, as the three-form $\Omega$ is holomorphic, it is not clear that it can always be normalized to be integral. Nekrasov notes, however, that the integrality condition is precisely the condition on the complex moduli of the CY to be solutions of the attractor equations. It would be interesting to understand if the integrality of $*\phi$ has a similar interpretation.

In [34] the authors want to consider solutions to the deformed Donaldson-Thomas equation

$$*\phi \wedge F = \frac{1}{6} F \wedge F \wedge F \, , \tag{6.32}$$

which would involve adding a term $CS_7(A)$ to the Lagrangian above. It is not at all clear why such a term would appear in OSFT but in section 7 we see that such a term does emerge in a rather interesting way when quantizing this theory.

### 6.6 Dimensional reduction, A- and B-branes

Reducing the open topological $G_2$ string on $CY_3 \times S^1$ gives rise to both special Lagrangian A-branes and holomorphic B-branes on $CY_3$. This follows from the decomposition of $\phi$ and $*\phi$ in terms of the holomorphic 3-form and Kähler form on $CY_3$ (see appendix A). The A-branes arise when reducing the associative 3-cycle action (B.21) in the normal direction. The resulting action

$$\int_M \epsilon^{abc} \mathrm{Tr}\left( A_a \nabla_b A_c + \frac{2}{3} A_a A_b A_c \right) + \rho^{aij} \mathrm{Tr}\big(\theta_i (\nabla_a \theta_j + [A_a, \theta_j])\big) \, , \tag{6.33}$$

is the real part of complex Chern-Simons theory, where the indices $a, b, c = 1, 2, 3$ are in the SLag while $i, j = 4, 5, 6$ are in the normal direction. The normal modes appear quadratically and can be integrated out (see section 7 for a discussion of this issue on an associated cycle).

Similarly we can reduce the 4-cycle action (6.23) in the tangential direction. This is again a special Lagrangian brane in $CY_3$ but now calibrated by $\hat{\rho}$ instead of $\rho$, and the worldvolume action is given by the imaginary part of complex Chern-Simons theory

$$\int_M \rho^{iab} \mathrm{Tr}(\theta_i F_{ab}) + \frac{2}{3} \rho^{ijk} \mathrm{Tr}(\theta_i \theta_j \theta_k) \, , \tag{6.34}$$

with the additional constraint $D_a \theta_i = 0$ for the normal modes.

The B-branes are simplest to find starting from the 7-cycle worldvolume theory (6.30) and reducing on the $CY_3$. We find

$$\begin{aligned} S &= \int_{CY_3} \hat{\rho} \wedge CS(A) + k \wedge k \wedge \mathrm{Tr}(\lambda F) \\ &= \frac{1}{2i} \int_{CY_3} \Omega \wedge CS(A) - \frac{1}{2i} \int_{CY_3} \bar{\Omega} \wedge CS(\bar{A}) + \int_{CY_3} k \wedge k \wedge \mathrm{Tr}(\lambda F) \, , \end{aligned} \tag{6.35}$$

where $*\phi = \hat{\rho} \wedge dt + \frac{1}{2} k \wedge k$, $t$ parametrizes the circle direction, $\Omega = \rho + i\hat{\rho}$ is the holomorphic 3-form of the Calabi-Yau, and $\lambda = A_t$ is the scalar component of the gauge field in the reduction. The action is the sum of B-model 6-brane and $\bar{\text{B}}$-model 6-brane actions (the appearance of the imaginary part of the holomorphic 3-form rather than the real part is just a matter of convention). The extra term in the action comes with the Lagrange multiplier $\lambda$, and so it expresses the constraint

$$k \wedge k \wedge F = 0 \,.$$

This extra condition is related to stability of the brane (complexifies the $\mathrm{U}(N)$ symmetry). Lower-dimensional 4-branes and 2-branes then follow by further dimensional reduction, where again we obtain B- and $\bar{\text{B}}$-model actions together with a stability condition.

It is remarkable that like the closed topological M-theory, the open topological string also contains the A and B+$\bar{\text{B}}$ models. Perturbatively the B+$\bar{\text{B}}$-models are decoupled, and it would be interesting to understand if there is a non-perturbative coupling between them.

## 7. Gauge-fixing and quantization

Let us now consider the full expansion of the OSFT action without any constraint on the ghost number of the fields. As found in appendix B.1, this gives the following expression for the action in seven dimensions

$$\begin{aligned} S_{(7)} &= \int_Y \phi^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho + \beta_{\mu\nu}\partial_\rho f + \beta_{\mu\nu}[A_\rho, f] + \frac{1}{2}C_{\mu\nu\rho}\{f,f\}\right) \\ &= \int_Y *\phi\wedge\mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A + \beta\wedge Df + \frac{1}{2}C\{f,f\}\right)\,, \end{aligned} \tag{7.1}$$

where $f\in\Lambda^0_{\mathbf{1}}$, $\beta\in\Lambda^2_{\mathbf{7}}$ and $C\in\Lambda^3_{\mathbf{1}}$ are respectively the degree zero, two and three modes of the string field $\mathcal{A}$ in the adjoint representation of the gauge group and $D = d + A$ is the gauge-covariant derivative. The purely bosonic (i.e. ghost number one field) part of the action above has appeared (in conjunction with additional bosonic terms) in topological quantum field theories studied in [35] and [36]. The interpretation of the action above in terms of the Batalin-Vilkovisky antifield formalism is detailed in appendix B.1.

### 7.1 Weak coupling limit

To help us understand the structure of the gauge theories we have found for open strings ending on (co)associative calibrated branes, we are more interested in the quantization of the quadratic part of the non-linear action $S[A] = \int_Y *\phi\wedge CS(A)$, expanded around solutions of the classical equations of motion

$$*\phi\wedge F = 0\,. \tag{7.2}$$

The partition function of this simplified theory corresponds to a stationary phase approximation of the full theory in the weak coupling limit. For the gauge theory on associative 3-cycles, we will investigate how the normal modes modify the corresponding calculation done by Witten [37] for pure Chern-Simons theory.

The equation $*\phi\wedge F = 0$ has been considered already in [19] where it is argued to be the 7-dimensional generalization of Chern-Simons theory that might provide an analog of Casson/Floer theory for 7-manifolds. It is related to an instanton equation for a gauge field on the Spin(7) 8-manifold $Y \times \mathbb{R}$. This relationship is directly analogous to the way solutions of the Chern-Simons equation of motion $F = 0$ on a 3-manifold $M$ correspond to critical points of the gradient flow equations coming from the instanton equations $F = *F$ on $M \times \mathbb{R}$. This fact will be important when we come to consider the non-trivial phase factor in the path integral of this gauge theory.

Expanding $S[A]$, for $A = A^0 + B$, to quadratic order in $B$ around a classical solution $A^0$ gives

$$S[A] = S[A^0] + \int_Y *\phi \wedge \mathrm{Tr}\,(B \wedge DB) \ , \tag{7.3}$$

where $D = d + [A^0, ]$ is here with respect to the background gauge field solving $\phi^{\mu\nu\rho} F^0_{\nu\rho} = 0$. The linear term is of course absent since it gives the $A^0$ equation of motion. Performing the BV analysis of the quadratic action $S_{cl}[B] = \int_Y *\phi \wedge \mathrm{Tr}(B \wedge DB)$ is straightforward and is given in appendix B.1 (it is also related to a linearization of the structure described for the full theory in appendix B.1).

The resulting gauge-fixed action takes the familiar form

$$\int_Y *\phi \wedge \mathrm{Tr}\,(B \wedge DB) + \mathrm{Tr}\,(\varphi D^\mu B_\mu + \bar{c} D^\mu D_\mu c) \ , \tag{7.4}$$

with $\varphi$ acting as Lagrange multiplier imposing the gauge-fixing constraint in the action while $\bar{c}$, $c$ correspond to the fermions from the Faddeev-Popov determinant.

Formally the analysis of this gauge theory in 7 dimensions has been almost identical to Witten's analysis of pure Chern-Simons in 3 dimensions. Indeed we can also use Schwarz's method of evaluating the partition function for degenerate quadratic classical actions to obtain the contribution

$$\exp(ikS[A^0]) \, \frac{\det(D_\mu D^\mu)}{\sqrt{\det(L)}} \ , \tag{7.5}$$

to the partition function of $ik \int_Y *\phi \wedge CS(A)$ (in the weak coupling limit of large $k$) coming from a given gauge-equivalence class of solutions $A^0$ of $*\phi \wedge F = 0$. We should stress that the structure of the moduli space of solutions to $*\phi \wedge F = 0$ is not understood so well as that for flat connections in 3 dimensions. Witten [37] restricts attention to Chern-Simons theory on 3-manifolds $M$ with the property that the moduli space of flat connections, determined by equivalence classes of homomorphisms from $\pi_1(M)$ to the gauge group $G$, be finite. We do not know whether one can take the moduli space of gauge-inequivalent solutions of $*\phi \wedge F = 0$ to be zero-dimensional by suitable choice of $G_2$ manifold $Y$. Thus we cannot say whether the partition function can be expressed as a finite sum over contributions of the form above.

The operator appearing in the denominator above is defined $L = *(*\phi \wedge D) + D*$ and is understood as an antisymmetric 8x8 matrix of linear differential operators mapping $\Lambda^1_{\mathbf{7}} \oplus \Lambda^7_{\mathbf{1}}$ to itself. It follows by collecting $B_\mu$ and $\varphi$ in the first two terms in the gauge-fixed quadratic action into an 8-vector. One can check that this definition implies $L$ is elliptic and self-adjoint. It seems the natural generalisation of the elliptic self-adjoint operator

$L_- = *D + D*$ (restricted to forms of odd degree in 3 dimensions) used by Witten in [37].[15] Another technical point we are overlooking is whether $L = *(*\phi \wedge D) + D*$ is a regular operator. We need not get into the precise definition, sufficed to say that regularity of an operator guarantees one has a precise definition of its determinant in terms of regularised zeta functions.

As explained in [37], the contribution to the partition function of Chern-Simons theory in 3 dimensions around a given flat connection at weak coupling is closely related to the partition function of an abelian 1-form gauge theory in 3 dimensions, which has been explicitly calculated by Schwarz and shown to give the Ray-Singer analytic torsion of the de Rham complex of the 3-manifold, and is thus a topological invariant. However, this relation to Ray-Singer torsion is generally only guaranteed for topological actions of the form $\int \omega \wedge d\omega$, where $\omega$ is a bosonic/fermionic $p$-form of odd/even degree in $(2p+1)$ dimensions. Thus we should not expect the partition function of the 7-dimensional quadratic theory above to be obviously related to Ray-Singer torsion. On the other hand, since we are still in odd dimension, a theorem of Schwarz [38] does suggest the partition function for this gauge theory should be a topological invariant. In fact this statement is only true modulo possible obstructions related to non-trivial phase factors that we will now discuss.

### 7.2 Phase of the determinant

An important subtlety in both 3 and 7 dimensions is the role of the phase of the determinant of the operator $L$. The theories described by Schwarz are insensitive to this since they compute absolute values of ratios of determinants of elliptic operators. The Laplacian $D_\mu D^\mu$ appearing in the numerator is real and positive-definite so there is no possible phase coming from its determinant. We will now investigate the structure of this phase for the 7-dimensional theory.

The expression for the phase in terms of the Atiyah-Patodi-Singer $\eta$-invariant follows in the same way in both 3 and 7 dimensions; as the limit of a series in powers of the non-zero eigenvalues $\lambda_i$ of the operator $L$ (at a given background solution $A^0$ of $*\phi \wedge F = 0$). In particular, as in [37], we find

$$\frac{1}{\sqrt{\det(L)}} = \frac{1}{|\sqrt{\det(L)}|} \exp\left(\frac{i\pi}{2}\eta_L(A^0)\right) , \tag{7.6}$$

where

$$\eta_L(A^0) = \frac{1}{2}\lim_{s\to 0}\sum_i \mathrm{sign}\lambda_i\, |\lambda_i|^{-s} , \tag{7.7}$$

denotes the $\eta$-invariant for the elliptic operator $L$ at solution $A^0$.

[15] In both dimensions 3 and 7, the addition of the $D*$ term in $L$ is essential in order for it to be elliptic. This is simply because without it the Pfaffian of the corresponding antisymmetric symbol matrices of odd rank would vanish identically and so there could exist no inverse. The way of understanding the need for ellipticity in physics terms is that we require the kinetic operator in the quadratic action to be the inverse propagator. The propagator only exists for the gauge-fixed action.

In 3 dimensions Witten [37] uses the Atiyah-Patodi-Singer index theorem for the classical twisted spin complex ($L_-$ can be interpreted as a twisted Dirac operator) to compute the difference of $\eta$-invariants between two flat connections, $A = A^0$ and $A = 0$, to be proportional to the Chern-Simons action $\int_M CS(A^0)$ itself at $A^0$. The proportionality factor is the dual Coxeter number $h(G)$ of the gauge group $G$. This has the beautiful interpretation of the level shift $k \to k + h(G)$ in the quantum Chern-Simons action, that one also observes for current algebras of conformal field theories in 2 dimensions.

The identification of $L_- = *D + D*$ in 3 dimensions with a twisted Dirac operator follows by collecting the differential operators in $L_-$ into a 4x4 antisymmetric matrix acting on the 4-dimensional vector space $\Lambda^1 \oplus \Lambda^3$. This allows one to write $L_- = \gamma^a D_a$ in terms of the 3 4x4 antisymmetric matrices $\gamma_a$, with components $(\gamma_a)_{bc} = -\epsilon_{abc}$, $(\gamma_a)_{b4} = -\delta_{ab}$. These matrices generate a subgroup $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ and in an appropriate basis can be written $\Gamma_a = i\sigma_2 \otimes \sigma_a$ (in terms of Pauli matrices $\sigma_a$). Together with $\Gamma_4 = i\sigma_1 \otimes 1$, they generate a representation of the Clifford algebra acting on Dirac spinors in 4 dimensions. By constructing the interpolating gauge field $A(t)$, for $t \in [0, 1]$ on $M \times [0, 1]$ between 2 flat gauge fields $A(1) = A^1$ and $A(0) = A^0$ on $M$, this provides a suitable lift of $L_-$ on $M$ to the twisted Dirac operator $\tilde{L}_- = \Gamma^a D_a(A(t)) + \Gamma^4 \partial_t$ on $M \times [0, 1]$. It is the Atiyah-Patodi-Singer index theorem for $\tilde{L}_-$ that allows Witten to compute the change in $\eta_{L_-}$ between 2 flat connections.

We will now show that a similar structure follows for $L = *(*\phi \wedge D) + D*$ in 7 dimensions. Again collecting the differential operators in $L$ into an 8x8 antisymmetric matrix acting on the 8-dimensional vector space $\Lambda^1 \oplus \Lambda^7$ allows one to express $L = \gamma^\mu D_\mu$ in terms of the 7 8x8 antisymmetric matrices $\gamma_\mu$, with components $(\gamma_\mu)_{\nu\rho} = -\phi_{\mu\nu\rho}$, $(\gamma_\mu)_{\nu 8} = -\delta_{\mu\nu}$. It should be noted that the sub-matrices $(\gamma_\mu)_{\nu\rho}$ do not form the adjoint representation of the imaginary octonions despite the fact that they are identical to the structure constants of this algebra. This is simply because the octonions are not associative. This is to be contrasted with the submatrices $(\gamma_a)_{bc}$ in 3 dimensions which give the adjoint representation of the imaginary quaternions (i.e. the Lie algebra of SU(2)). Nonetheless, together with $\gamma_8 = i1$, the full 8x8 matrices $\gamma_\mu$ generate a representation of the Clifford algebra acting on Weyl spinors in 8 dimensions. The corresponding action on Dirac spinors in 8 dimensions can be expressed in terms of the 16x16 anti-Hermitian matrices $\Gamma_\mu = \sigma_2 \otimes \gamma_\mu$, $\Gamma_8 = i\sigma_1 \otimes 1$. Thus by constructing the interpolating gauge field $A(t)$ on $Y \times [0, 1]$ between 2 solutions $A(1) = A^1$ and $A(0) = A^0$ of $*\phi \wedge F = 0$ on $Y$ we have a suitable lift of $L$ on the $G_2$ manifold $Y$ to the twisted Dirac operator $\tilde{L} = \Gamma^\mu D_\mu(A(t)) + \Gamma^8 \partial_t$ on $Y \times [0, 1]$.

Before obtaining the change in $\eta_L$ from the Atiyah-Patodi-Singer index theorem for $\tilde{L}$, it may be illuminating to make a brief digression explaining how this lift of $L$ is related to the elliptic complex

$$0 \longrightarrow \mathrm{ad}\, G \otimes \Lambda^0 \xrightarrow{D} \mathrm{ad}\, G \otimes \Lambda^1 \xrightarrow{\frac{1}{4}(1-*\Psi\wedge)D} \mathrm{ad}\, G \otimes \Lambda^2_{\mathbf{7}} \longrightarrow 0\,, \tag{7.8}$$

on an 8-manifold $X$ of Spin(7) holonomy, with Cayley 4-form $\Psi = *\Psi$, when $X = Y \times [0, 1]$. This complex has been used in the study of 8-dimensional topological quantum field theories in [35]. The operator $\pi^2_{\mathbf{7}} = \frac{1}{4}(1 - *\Psi\wedge)$ projects the 2-form in 8 dimensions onto the 7-dimensional irreducible representation of Spin(7). The adjoint operators mapping to the left of the complex are $D^\dagger$. As noted by Donaldson and Thomas, solutions of $*\phi\wedge F=0$ on the $G_2$ manifold $Y$ correspond to fixed points of the gradient flow from the Spin(7) instanton equation $*F=\Psi\wedge F$ on $Y\times\mathbb{R}$ (i.e. elements of the kernel of $\pi^2_{\mathbf{7}}D$).

The relation of this complex to the twisted spin complex for $\tilde{L}$ follows by observing the isomorphisms $\mathsf{S}_+ = \Lambda^0_{\mathbf{1}}\oplus\Lambda^2_{\mathbf{7}}$ and $\mathsf{S}_- = \Lambda^1_{\mathbf{8}}$ for the positive and negative chirality spin bundles $\mathsf{S}_\pm$ on a Spin(7) manifold (using the conventions of [36] where the Spin(7)-invariant spinor $\theta\in\mathsf{S}_+$). The explicit isomorphisms following from Fierz identities give $\psi_+=\eta\theta-\frac{1}{4}\chi_{MN}\Gamma^{MN}\theta$ and $\psi_-=-\psi_M\Gamma^M\theta$ $(M,N=1,\dots,8)$ for any $\psi_\pm\in\mathsf{S}_\pm$, where $\eta=\theta^t\psi_+$ is a scalar, $\chi_{MN}=\frac{1}{2}\theta^t\Gamma_{MN}\psi_+$ is a 2-form obeying the identity $\pi^2_{\mathbf{7}}\chi=\chi$ and $\psi_M=\theta^t\Gamma_M\psi_-$ is a 1-form. The action of the twisted Dirac operator $\Gamma^M D_M:\mathsf{S}_-\to\mathsf{S}_+$ on these expressions gives $\Gamma^M D_M\psi_-=(D^M\psi_M)\theta-(\pi^2_{\mathbf{7}}D\psi)^{MN}\Gamma_{MN}\theta$ hence equating $\Gamma^M D_M$ acting on $\mathsf{S}_-$ with $\pi^2_{\mathbf{7}}D+D^\dagger$ acting on $\Lambda^1_{\mathbf{8}}$ in the complex above. This is consistent with the reduction of the lifted $\tilde{L}$ on $Y\times[0,1]$ to $L=*(*\phi\wedge D)+D*$ on $Y$. Using this identification, one can check that the index of the whole Spin(7) complex above is identical to that for the twisted Dirac operator on a Spin(7) manifold.

This identification has been used by Reyes-Carrión [23] to calculate the Atiyah-Singer index

$$\int_X ch(\mathrm{ad}G)\hat{A}(TX)=\int_X \dim(G)\hat{A}_2(TX)+\frac{1}{24}\left(p_1(TX)\wedge c_2(\mathrm{ad}G)+2(c_2(\mathrm{ad}G))^2-4c_4(\mathrm{ad}G)\right), \tag{7.9}$$

of the Spin(7) complex above, on a closed Spin(7) 8-manifold $X$. The A-roof genus $\int_X\hat{A}_2$ here corresponds to the number of parallel spinors on $X$ and so equals 1 if the holonomy is exactly Spin(7) (and not a subgroup thereof). For convenience, it is assumed in the formula above that the gauge group is chosen such that the Chern classes $c_1(\mathrm{ad}\,G)$ and $c_3(\mathrm{ad}\,G)$ both vanish (e.g. for $G=\mathrm{SU}(N)$).

Consider now $X=Y\times[0,1]$ where $A(t)$ interpolates between two solutions $A=A^0$ and $A=0$ of $*\phi\wedge F=0$ on $Y$. The Atiyah-Patodi-Singer index theorem for $\tilde{L}$ is

$$\mathrm{ind}(\tilde{L})=\int_{Y\times[0,1]} ch(\mathrm{ad}\,G)\hat{A}(T(Y\times[0,1]))-\frac{1}{2}[\eta_L(A^0)-\eta_L(0)]\,. \tag{7.10}$$

The bulk integral can be evaluated using the Reyes-Carrión result on $X=Y\times[0,1]$. This is equal to the continuous part of $\frac{1}{2}[\eta_L(A^0)-\eta_L(0)]$ and is given by

$$\dim(G)+\frac{1}{24}\left(\frac{1}{2\pi}\right)^4\int_Y\left[-\frac{1}{2}\mathrm{Tr}(R\wedge R)\wedge CS_3(A^0)+CS_7(A^0)\right]\,, \tag{7.11}$$

as an integral over the $G_2$ manifold $Y$ with Riemann curvature $R$. The Chern-Simons forms are

$$\begin{aligned}CS_3(A)&=\mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A^3\right)\,,\\ CS_7(A)&=\mathrm{Tr}\left(A\wedge(dA)^3+\frac{16}{5}A^3\wedge(dA)^2+\frac{4}{5}A^2\wedge dA\wedge A\wedge dA+2\,A^5\wedge dA+\frac{4}{7}A^7\right)\,.\end{aligned} \tag{7.12}$$

In general $\frac{1}{2}[\eta_L(A^0) - \eta_L(0)]$ can also have a discontinuous contribution, corresponding to the spectral flow of $L$, and is equal to (minus) the index of the lifted operator $\tilde{L}$ itself. This has the effect of shifting the continuous part of $\frac{1}{2}[\eta_L(A^0) - \eta_L(0)]$ by $\pm 1$ if the eigenvalues $\lambda_i(t)$ of $L(A(t))$ (understood as a function of $t$) change sign when $t$ is varied between 0 and 1 (a $+1$ shift corresponds to a change $\lambda_i < 0$ to $\lambda_i > 0$).

The addition of 'constant' terms (that do not depend on the particular choice of solutions $A^1$ and $A^0$) to $\frac{1}{2}[\eta(A^1) - \eta(A^0)]$ will have a trivial effect that can be factored out of the overall phase structure of the theory and ignored. Thus the effect of the spectral flow of a given operator can only be ignored if it is a constant in this sense. This is the case for Witten's analysis of $L_-$ in 3 dimensions. This is obviously also true for the constant $\dim(G)$ in the change in the $\eta$-invariant above. It is not clear to us whether the effect of the spectral flow of $L$ in 7 dimensions will be significant and we will overlook this subtlety here.

Therefore it is clear that the phase structure of the 7-dimensional theory is much more complicated than just the level shift that occurs in 3 dimensions. Nonetheless, let us examine some of the terms in $\frac{1}{2}[\eta_L(A^0) - \eta_L(0)]$ in a bit more detail.

The term $\mathrm{Tr}(R \wedge R)$, proportional to the first Pontrjagin class of $Y$, which ordinarily can be a general element of $H^4(Y, \mathbb{R})$, is here somewhat constrained due to the fact that $Y$ must have holonomy in $G_2$. In particular, this constrains the curvature such that $\pi^2_{\mathbf{7}} R = 0$ (or $R_{\mu\nu\alpha\beta}\phi^{\alpha\beta\gamma} = 0$ in components) so that the holonomy algebra is contained in the Lie algebra of $G_2$. Decomposing

$$H^4(Y, \mathbb{R}) = H^4_{\mathbf{1}}(Y, \mathbb{R}) \oplus H^4_{\mathbf{7}}(Y, \mathbb{R}) \oplus H^4_{\mathbf{27}}(Y, \mathbb{R}) \,, \tag{7.13}$$

into irreducible representations of $G_2$, one can check that the constraint above implies $\mathrm{Tr}(R \wedge R)$ has no **7** part. (This also follows from lemma 1.1.2 in [39], although only compact manifolds with full $G_2$ holonomy are considered there and so one has the stronger constraint $b^4_{\mathbf{7}} = 0$ which we need not assume here.)

The cohomology group $H^4_{\mathbf{1}}(Y, \mathbb{R}) = \mathbb{R}$ has a very simple structure, being spanned by constant multiples of the harmonic 4-form $*\phi$. Moreover, one can prove the identity $\mathrm{Tr}(R \wedge R) \wedge \phi = -|R|^2 \,\mathrm{vol}$ implying the constant multiplying the **1** part of the first Pontrjagin class is negative definite and vanishes only if the $G_2$ metric is flat (this also follows from lemma 1.1.2 in [39]). Hence the contribution to the expression for $\eta$ above coming from this term will cause a positive shift in the effective coupling constant $k$ for the action $\int_Y *\phi \wedge CS(A^0)$, reminiscent of the level shift in 3-dimensional Chern-Simons theory.

The final contribution to the first Pontrjagin class coming from $H^4_{\mathbf{27}}(Y, \mathbb{R})$ is more complicated and generally will not vanish. Recall it is precisely elements of $H^3_{\mathbf{27}}(Y, \mathbb{R}) = H^4_{\mathbf{27}}(Y, \mathbb{R})$ that parameterize deformations of a given $G_2$ manifold such that the deformed manifold is also $G_2$. Hence this contribution would vanish for 'rigid' $G_2$ manifolds with no deformation moduli (or, of course, for special $G_2$ manifolds whose first Pontrjagin class has no **27** part).

The effect on the partition function from the contribution to $\eta_L$ from $CS_7(A^0)$ is also rather complicated. We will simply note that the equations of motion arising from a modification to the classical action of this kind would be of the form

$$*\phi \wedge F = \lambda F \wedge F \wedge F \ , \tag{7.14}$$

for some constant $\lambda$, which were considered by Leung et al as a deformed version of Donaldson-Thomas theory.

Just as in 3 dimensions, we expect that the overall $\eta_L(0)$ exponential prefactor in the partition function will not be a topological invariant. The task of finding a different regularisation that preserves general covariance is much more difficult in 7 dimensions and we will not attempt this here.

### 7.3 3-cycle worldvolume theory

Let us now repeat the analysis of the previous section as far as possible to describe the quantization of the 3-cycle theory. The effective action for this theory

$$\begin{aligned} S_{(3)} = \int_M \epsilon^{abc}\mathrm{Tr}\left(A_a\partial_b A_c + \frac{2}{3}A_a A_b A_c + \beta_{ab}D_c f + \frac{1}{2}C_{abc}[f,f]\right) \\ + \phi^{aIJ}\mathrm{Tr}\left(\theta_I D_a \theta_J + 2\,\beta_{aI}[\theta_J, f]\right) \ , \end{aligned} \tag{7.15}$$

(derived from OSFT in appendix B.2) is essentially pure Chern-Simons theory for the gauge field $A_a$ on $M$, which is a completely solvable theory, plus additional normal mode contributions from $\theta_I$, whose effect we shall investigate ($D_a = \nabla_a + [A_a, -]$ on $M$). It can also be understood as the dimensional reduction of the 7-cycle action $S_{(7)}$ (after appropriately rescaling $\beta$ and $C$).

In principle a similar modification by normal modes may occur for open strings ending on special Lagrangian 3-cycles in Calabi-Yau manifolds in the A-model, though this is not discussed in [13]. There is considerable evidence, however, that the worldvolume theory on a special Lagrangian is essentially just Chern-Simons theory (up to possible worldsheet instanton corrections), as this is used, for instance, in open-closed transitions [40]. An essential point is that, aside from $A_a$, none of the other fields in the 3-cycle action appears at higher than quadratic order in the Lagrangian so they can be integrated out exactly.

#### 7.3.1 1-loop partition function

Let us again simplify matters by quantizing the quadratic part of the non-linear action $S_{(3)}$, expanded around solutions of the equations of motion (6.21), (6.22) for the classical part $S[A,\theta] = \int_M CS(A) + \phi^{aIJ}\mathrm{Tr}(\theta_I D_a \theta_J)$ of $S_{(3)}$.

Expanding $S[A,\theta]$, for $A_a = A_a^0 + B_a$, $\theta_I = \theta_I^0 + \xi_I$, to quadratic order in $(B,\xi)$, around a classical solution $(A^0, \theta^0)$ gives

$$S[A,\theta] = S[A^0,\theta^0] + \int_M \epsilon^{abc}\mathrm{Tr}\left(B_a D_b B_c\right) + \phi^{aIJ}\mathrm{Tr}\left(\xi_I D_a \xi_J + \theta_I^0[B_a, \xi_J]\right) \ , \tag{7.16}$$

where $D_a = \nabla_a + [A_a^0, -]$. The BV structure of the quadratic action

$$S_{cl}[B,\xi] = \int_M \epsilon^{abc}\mathrm{Tr}(B_a D_b B_c) + \phi^{aIJ}\mathrm{Tr}(\xi_I D_a \xi_J + \theta_I^0[B_a, \xi_J]) \ , \tag{7.17}$$

is detailed in appendix B.2. The resulting gauge-fixed action takes the expected form

$$S_{cl}[B,\xi] + \int_M \mathrm{Tr}\,(\varphi D_a B^a + \bar{c} D_a D^a c)\ . \tag{7.18}$$

To compare this quantum theory with Witten's analysis of pure Chern-Simons theory, let us begin by calculating the contribution to the path integral from a flat connection (i.e. $A^0$ is flat and $\theta^0 = 0$, solving (6.21) and (6.22)). The modification to equation (2.8) of [37] (for the contribution from a flat connection $A^0$ in pure Chern-Simons theory) due to the normal modes is given by

$$\mu(A^0, 0) = \exp(ikS[A^0, 0])\, \frac{\det(D_a D^a)}{\sqrt{\det(L_- \oplus \phi^{aIJ} D_a)}}\ , \tag{7.19}$$

where $\phi^{aIJ} D_a$ is understood as a 4x4 antisymmetric matrix of differential operators. We can go further by making use of the important identity $\phi^{aIJ} D_a \phi^{bJK} D_b = -\delta^{IK} D_a D^a$, which follows using $F^0_{ab} = 0$. This is related to the fact that $\phi^{aIJ}$, understood as 3 4x4 matrices, generate an SU(2) subgroup of the SO(4) structure group of the normal bundle of $M$ and can be understood as Pauli matrices. This allows us to identify $\phi^{aIJ} D_a$ as a twisted Dirac operator acting on a 4-dimensional vector space, just as Witten did for $L_-$. Hence going through the usual Atiyah-Patodi-Singer analysis of the phase factor for the direct sum of two identical twisted spin complexes over $M \times [0,1]$ (both twisted by $A^0$) implies the difference of $\eta$-invariants between 2 flat connections $A = A^0$ and $A = 0$ will also be proportional to the pure Chern-Simons action at $A^0$. Hence this will give essentially the same 1-loop effective action as for pure Chern-Simons theory except the shift in the level will effectively be doubled.

Understanding the effect of contributions from more general solutions of (6.21) and (6.22) is a more difficult task since not much is known about this moduli space other than that it contains flat connections. Formally the contribution from a general solution $\mu(A^0, \theta^0)$ will be similar to $\mu(A^0, 0)$ but for replacing $S[A^0, 0]$ by $S[A^0, \theta^0]$ in the exponential and including an off-block-diagonal component $\phi^{aIJ}\theta^0_J$ for the determinant in the denominator. It may prove more convenient to understand such contributions from the 7-dimensional perspective.

### 7.4 4-cycle worldvolume theory

The action (6.23) also follows from reduction of the 7-dimensional action $\int_Y *\phi \wedge CS(A)$ on the 4-cycle. The ghost structure of this theory is derived from OSFT in appendix B.2, just as for the 3-cycle theory, and again follows from dimensional reduction of the 7-dimensional theory (up to suitable field re-scalings) to give the full 4-cycle action

$$\begin{aligned} S_{(4)} = \int_M \phi^{Iab} \mathrm{Tr}\big(\theta_I F_{ab}\big) + \frac{2}{3}\,\phi^{IJK} \mathrm{Tr}\big(\theta_I \theta_J \theta_K\big) + \frac{1}{2} \phi^{IJK} \mathrm{Tr}\big(C_{IJK}[f,f]\big) \\ + 2\,\phi^{Iab} \mathrm{Tr}\big(\beta_{Ia} D_b f\big) + \phi^{IJK} \mathrm{Tr}\big(\beta_{IJ}[\theta_K, f]\big)\ . \end{aligned} \tag{7.20}$$

#### 7.4.1 1-loop partition function

Proceeding as in the previous sections, we quantize the quadratic part of $S_{(4)}$ by expanding around solutions of (6.24), (6.27) for the classical part of $S_{(4)}$

$$S[A,\theta] = \int_M \phi^{Iab}\mathrm{Tr}\big(\theta_I F_{ab}\big) + \frac{2}{3}\phi^{IJK}\mathrm{Tr}\big(\theta_I\theta_J\theta_K\big)\ . \tag{7.21}$$

Expanding $S[A,\theta]$, for $A_a = A_a^0 + B_a$, $\theta_I = \theta_I^0 + \xi_I$, to quadratic order in $(B,\xi)$, around a classical solution $(A^0,\theta^0)$ gives

$$S[A,\theta] = S[A^0,\theta^0] + 2\int_M \phi^{Iab}\mathrm{Tr}\big(\xi_I D_a B_b + \theta_I^0 B_a B_b\big) + \phi^{IJK}\mathrm{Tr}\big(\theta_I^0\xi_J\xi_K\big)\ , \tag{7.22}$$

where $D_a = \nabla_a + [A_a^0, -]$. The BV analysis of the quadratic action

$$S_{cl}[B,\xi] = \int_M \phi^{Iab}\mathrm{Tr}\big(\xi_I D_a B_b + \theta_I^0 B_a B_b\big) + \phi^{IJK}\mathrm{Tr}\big(\theta_I^0\xi_J\xi_K\big) \tag{7.23}$$

is given in appendix B.2, leading to the expected gauge-fixed action

$$S_{cl}[B,\xi] + \int_M \mathrm{Tr}\left(\varphi D_a B^a + \bar{c} D_a D^a c\right)\ . \tag{7.24}$$

We will now begin to analyse the quantum structure of this theory by calculating the contribution to the path integral from an instanton configuration (i.e. $A^0$ obeys $\phi^{Iab}F_{ab} = 0$ and $\theta^0 = 0$, solving (6.24) and (6.27)). The contribution is given by

$$\mu(A^0, 0) = \frac{\det(D_a D^a)}{\det(\phi^{abI}D_b \oplus D*)}\ , \tag{7.25}$$

where $\phi^{abI}D_b$ is understood as a 4x3 matrix of differential operators which, together with $D*$ acting on 4-forms, makes up a square 4x4 antisymmetric matrix that provides an involutive mapping $\Lambda^0(NM)\oplus\Lambda^4(M)\to\Lambda^1(M)$. The reason there is no square root in the denominator is that the differential operator appearing in the gauge-fixed action is an 8x8 matrix (acting on $B_a$, $\xi_I$ and $\varphi$) with zeros in the 4x4 block-diagonal entries and the 4x4 operators above in both off-block-diagonal entries. It is not clear to us if this determinant can be simplified further or whether it contributes a non-trivial phase factor. The structure of $\theta_I^0 \neq 0$ contributions is also unclear.

## 8. Remarks and open problems

So far in this paper we have determined the spectrum of the open $G_2$ string and related it to the worldvolume field theories of branes in a $G_2$ manifold. In this section we would like to conclude by making some final remarks regarding issues that still need to be resolved as well as interesting directions for further research.



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## 8.1 Holomorphic instantons on special Lagrangians

In dimensionally reducing the $G_2$ branes on a Calabi-Yau $Z$ times a circle, we have found that we almost reproduce the real versions of the gauge theories for the open A- and B-models. There is a discrepancy, however. If one considers a special Lagrangian $M \subset Z$, with holomorphic open curves $\Sigma \subset Z$ ending on $M$ so that $\partial\Sigma \subset M$, then the A-model branes will receive worldsheet instanton corrections to the standard Chern-Simons action. A naive dimensional reduction of the associative theory on a $G_2$ manifold $Y = Z \times S^1$ gives a special Lagrangian in $Z$ with the Chern-Simons action without instanton corrections.

This issue is already present in the closed topological $G_2$ string. When reducing on $CY_3 \times S^1$, the closed $G_2$ string gives a combination of A and B+$\bar{\text{B}}$ models. But it is non-trivial to see where the worldsheet instanton corrections in the A-model would come from, given that the $G_2$ theory appears to localize on constant maps. A possible resolution suggested in [8] is that since, unlike a generic $G_2$ manifold, the manifold $CY_3 \times S^1$ has 2-cycles, worldsheet instantons may now wrap these 2-cycles. However, upon closer inspection, this possibility appears rather unlikely. A much more straightforward explanation is that the worldsheet instanton contribution is due to topological membranes (i.e. topological 3-branes of the type discussed in this paper) that wrap associative cycles of the form $\Sigma \times S^1$ in $CY_3 \times S^1$. Such 3-cycles are indeed associative as long as $\Sigma$ is a holomorphic curve in the Calabi-Yau manifold.

Returning to the open worldsheet instanton contribution to branes in the A-model, there are two ways to obtain these from the topological $G_2$ string on $CY_3 \times S^1$. The first way is to lift the A-model brane together with the open worldsheet instanton to a single associative cycle in $CY_3 \times S^1$. This is similar to the M-theory lift in terms of a single M2-brane of a configuration of a fundamental string ending on a D2-brane in type IIA string theory. To describe it, we take a special Lagrangian 3-cycle $C$ in a Calabi-Yau manifold $X$, plus an open holomorphic curve $\Sigma$. We denote the boundary of $\Sigma$ by $\gamma \subset C$. We first lift $C$ to $X \times S^1$, which we describe in terms of a map $C \to X \times S^1$ which takes $x \in C$ to $(x, \theta(x)) \in X \times S^1$. Here, $\theta(x)$ describes an $S^1$-valued function on $C$ which we want to have the property that it winds once around the $S^1$ as we wind once around the curve $\gamma \subset C$. The lift is therefore one-to-many, as the image of a point in $\gamma$ is an entire circle, and because of this the lift of $C$ is an open submanifold of $X \times S^1$ with boundary $\gamma \times S^1$. We can now glue the naive lift of $\Sigma$, which is $\Sigma \times S^1$, to the lift of $C$ to form a closed 3-manifold $M$, since the boundary of $\Sigma \times S^1$ is also $\gamma \times S^1$. In this way we have obtained a closed 3-manifold $M \subset X \times S^1$ which projects down to $C$ and $\Sigma$ upon reduction over the $S^1$. The 3-manifold $M$ is not calibrated, but we can compute the integral of $\phi$ over $M$. The result is simply $\int_C \rho + \int_\Sigma k$ if we normalize the size of the $S^1$ appropriately. The fact that the lift of $C$ winds around the circle does not yield any additional contribution to $\int_M \phi$ because the restriction of $k$ to $C$ vanishes identically.

We have thus constructed a closed 3-cycle $M$ such that the integral of $\phi$ over it has the correct structure, geometrically, to yield the worldsheet instanton contribution. The final step is to minimize the volume of $M$ while keeping its homology class fixed. This will not change $\int_M \phi$ but presumably lead to the sought-for associative 3-cycle with the right properties.

In order to push this program further and relate $\int_\Sigma k$ to the (exponentiated) weight of a holomorphic instanton we note that maps $\theta(x)$ which wind about $\gamma$ $n$ times will generate contributions such as $n\int_\Sigma k$. Carefully summing over all lifts of this form with the appropriate weight might properly reproduce the instanton contributions.

An entirely alternative approach is to lift both $C$ and $\Sigma$ to $C \times S^1$ and $\Sigma \times S^1$. In this way we obtain an open associative 3-cycle ending on a coassociative 4-cycle in $X \times S^1$. To analyze whether this makes sense, we consider the simple example of an open 3-brane in $\mathbb{R}^7$ stretched along the 123-direction, ending on a coassociative cycle stretching in the 2345-direction. If we vary the action (6.20) on the 3-brane we obtain a boundary term

$$S_{\text{boundary}} = \int dx^2 dx^3 \text{tr}(A_3\delta A_2 - A_2\delta A_3 + \theta_5\delta\theta_4 - \theta_4\delta\theta_5 + \theta_7\delta\theta_6 - \theta_6\delta\theta_7)\ . \tag{8.1}$$

We obviously want Dirichlet boundary conditions for $\theta_6$ and $\theta_7$ so that the endpoint of the open 3-brane is confined to lie in the 4-brane. We also want $\theta_4$ and $\theta_5$ to be unconstrained at the boundary. If we therefore choose the boundary condition

$$A_2 = \theta_5 \qquad A_3 = \theta_4\ , \tag{8.2}$$

the variations all cancel. To preserve these boundary conditions under a gauge transformation, we need to restrict the gauge parameter in such a way that its derivatives in the $2, 3$ vanish at the boundary. In this way we indeed find a consistent open 3-brane ending on a 4-brane.

### 8.2 Extensions

The actions we have discovered on topological branes wrapping cycles in a $G_2$ manifold are variants of Chern-Simons theories derived from OSFT. OSFT itself, as a generator of perturbative string amplitudes, might need to be augmented by terms that are locally BRST trivial but none-the-less have global meaning deriving from the topological structure of the space of string fields. In the bosonic open string such questions are currently inaccessible but in the topological case we see some motivation for local total derivative terms to be added to the action. One such potential term is

$$\int_Y F \wedge F \wedge \phi \tag{8.3}$$

that might describe lower dimensional branes dissolved in the seven dimensional brane. Such terms might be motivated by analogy with the Wess-Zumino terms on physical branes. Note, also, that this reduces to $F \wedge F \wedge k$ in six-dimensions, a term which appears in the A-model Kähler quantum foam theory [41] which Nekrasov suggests should be related to holomorphic Chern-Simons theory [2] (the latter is, of course, related to our theory by dimensional reduction). It would be interesting to try and probe for the existence of such terms directly in the $G_2$ world-sheet or OSFT theory.

The appearance of the $CS_7(A)$ term in the one-loop partition function suggests that perhaps this term appears in quantizing the theory and so should have been included in the original classical action.

Understanding if such terms do actually appear in these effective actions is interesting as it may play a role in the conjectured S-duality of the A/B model topological strings. In the latter it seems that one may need to consider both the open and closed theory simultaneously and then terms such as (8.3) might play a role in coupling these theories.

### 8.3 Relation to twists of super Yang-Mills

The theories we have found on $G_2$ branes are all topological theories of the Schwarz type (see [30] for the terminology) which is no doubt linked to the fact that they are generated by OSFT. A similar statement holds for branes in the A- and B-model.

The worldvolume theory on a brane in a $G_2$ or Calabi-Yau manifold in a physical model is a twisted, dimensionally reduced super Yang-Mills (SYM) theory [42] whose ground states are topological in nature. These are related to the topological field theories that can be constructed by twisting SYM and considering only the supersymmetric states (by promoting the twisted supercharge to a BRST operator). Such theories include the topological action for Donaldson-Witten theory [43] as well as its generalizations to higher dimensions [35]. These are generally field theories of the Witten type meaning that the action is itself a BRST commutator plus a locally trivial term.

Aside from the obvious connection to Chern-Simons theory via OSFT it would be interesting to understand why the topological theories on branes in topological string theory are generally of the Schwarz type (which are locally non-trivial) while the supersymmetric states of the twisted theories on a physical brane can be studied in a theory that is of the Witten type.

### 8.4 Geometric invariants

One of the most interesting open directions is to investigate the geometric or topological invariants our open worldvolume gauge theories compute, and perhaps use them, via open-closed duality, to discover the connection to the closed topological $G_2$ theory. It would be interesting to explore the full quantum open string partition function on a few examples of $G_2$ manifolds. The theory on the 3-cycle is basically Chern-Simons theory, while on the 4-cycle the gauge theory of ASD connections will be related naturally to Donaldson theory. It would very interesting to find a role for the partition functions in terms of the full physical string theory, as well as deepen connections with the mathematics results in [31]. Another open problem is to analyze these invariants in the special case of $CY_3 \times S^1$, and find a physical understanding of related mathematical invariants such as the one proposed by Joyce [44] counting special Lagrangian cycles in a Calabi-Yau manifold.

### 8.5 Geometric transitions

Open-closed duality techniques have proven very useful for topological string theory on Calabi-Yau manifolds. In particular, geometric transitions provide nice examples where closed topological string amplitudes can be computed from the gauge theory on the branes, which in this case is just Chern-Simons theory with possible worldsheet instanton corrections. Geometric transitions on $G_2$ manifolds in general are less studied, but interesting examples from the full string theory point of view are exhibited in e.g. [45, 46]. In the present paper we derived the relevant worldvolume gauge theory actions from open topological strings and so, one of the immediate applications of our results is to study geometric transitions from the topological $G_2$ string point of view.

### 8.6 Mirror symmetry for $G_2$

Mirror symmetry on a Calabi-Yau 3-fold can be described in terms of the Strominger-Yau-Zaslow (SYZ) conjecture. One starts with a special Lagrangian fibration, and then the mirror manifold is conjectured to be the dual torus fibration over the same base. In physics language, the action of mirror symmetry on the fibres is T-duality. In [34], a $G_2$ version of the SYZ conjecture was suggested, relating coassociative to associative geometry. Evidence for the $G_2$ mirror symmetry was also found in $G_2$ compactifications of the physical IIA/IIB string theory on $G_2$ holonomy manifolds [46, 47]. It would be interesting to explore the action of mirror symmetry in the case of the topological $G_2$ models. A good starting point for this is by examining automorphisms of the closed $G_2$ string algebra such as those discussed in [48].

### 8.7 Zero Branes

Although we have not attempted a treatment here it should be possible to reduce the action (6.30) to zero dimensions to determine the world-volume of $D0$-branes on the $G_2$ manifold. This will be a matrix model which may be related in an interesting way to the $G_2$ geometry.

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## A. Conventions

In this section we will detail the conventions used in dealing with the associative 3-form and coassociative 4-form on a 7-manifold with $G_2$ holonomy. We adopt the conventions of [8] since we use many results from that paper. More details and original references for $G_2$ holonomy manifolds can be found in that paper.

Although we will generally not have need for the explicit form of $\phi$ or $*\phi$ we provide a definition in terms of local coordinates, using the conventions of [8]

$$\phi = \omega^{123} + \omega^1 \wedge (\omega^{45} + \omega^{67}) + \omega^2 \wedge (\omega^{46} - \omega^{57}) - \omega^3 \wedge (\omega^{47} + \omega^{56}) \,, \tag{A.1}$$

$$*\phi = \omega^{4567} + \omega^{23} \wedge (\omega^{67} + \omega^{45}) + \omega^{13} \wedge (\omega^{57} - \omega^{46}) - \omega^{12} \wedge (\omega^{56} + \omega^{47}) \,, \tag{A.2}$$

where $\omega^i$ are vielbeins and $\omega^{ij} = \omega^i \wedge \omega^j$ etc.

We also reproduce some identities for $\phi$ and $*\phi$ from [8] that we will have need of. The precise factors in these identities depends on a choice of conventions and normalizations (e.g. their normalizations are related to those used in [5] by $\phi^{\text{here}}_{\mu\nu\rho} = \frac{1}{3!}\phi^{\text{there}}_{MNP}$ and $*\phi^{\text{here}}_{\mu\nu\rho\sigma} = \frac{1}{4!}*\phi^{\text{there}}_{MNPQ}$).

$$\begin{aligned}
\phi^{\mu\alpha\beta}\phi_{\alpha\beta\nu} &= \frac{1}{6}\delta^\mu_\nu \ , \\
(*\phi)_{\mu\nu\alpha\beta}\phi^{\alpha\beta\gamma} &= \frac{1}{6}\phi_{\mu\nu}{}^\gamma \ , \\
\phi_{\mu\nu\gamma}\phi^{\gamma\alpha\beta} &= \frac{2}{3}(*\phi)_{\mu\nu}{}^{\alpha\beta} + \frac{1}{18}\delta^\alpha_{[\mu}\delta^\beta_{\nu]} \ , \\
(*\phi)_{\mu\nu\gamma\rho}(*\phi)^{\gamma\rho\alpha\beta} &= \frac{1}{12}(*\phi)_{\mu\nu}{}^{\alpha\beta} + \frac{1}{72}\delta^\alpha_{[\mu}\delta^\beta_{\nu]} \ .
\end{aligned} \tag{A.3}$$

The exterior algebra on a $G_2$ manifold can be decomposed into irreducible representations of $G_2$. The decomposition is given as follows

$$\begin{aligned}
\Lambda^0 &= \Lambda^0_{\mathbf{1}} \ , & \Lambda^1 &= \Lambda^1_{\mathbf{7}} \ , \\
\Lambda^2 &= \Lambda^2_{\mathbf{7}} \oplus \Lambda^2_{\mathbf{14}} \ , & \Lambda^3 &= \Lambda^3_{\mathbf{1}} \oplus \Lambda^3_{\mathbf{7}} \oplus \Lambda^3_{\mathbf{27}} \ .
\end{aligned} \tag{A.4}$$

Subscripts here indicate the dimension of the irreducible representation of $G_2$. The decomposition of higher degree forms follows by Hodge duality $*\Lambda^i_{\mathbf{n}} = \Lambda^{7-i}_{\mathbf{n}}$.

We will frequently have use for the explicit form of the projectors onto these representations

$$\begin{aligned}
(\pi^2_{\mathbf{7}})_{\mu\nu}{}^{\alpha\beta} &= 6\phi_{\mu\nu\gamma}\phi^{\gamma\alpha\beta} \ , \\
(\pi^2_{\mathbf{14}})_{\mu\nu}{}^{\alpha\beta} &= -4(*\phi)_{\mu\nu}{}^{\alpha\beta} + \frac{2}{3}\delta^\alpha_{[\mu}\delta^\beta_{\nu]} \ , \\
(\pi^3_{\mathbf{1}})_{\mu\nu\rho}{}^{\alpha\beta\gamma} &= \frac{1}{7}\phi_{\mu\nu\rho}\phi^{\alpha\beta\gamma} \ .
\end{aligned} \tag{A.5}$$

When a $G_2$ manifold has the structure $CY_3 \times S^1$, there is a decomposition of $\phi$ and $*\phi$ in terms of $\rho = \mathrm{Re}(e^{i\alpha}\Omega)$, $\hat\rho = \mathrm{Im}(e^{i\alpha}\Omega)$ and $k$ (where $\Omega$ is the holomorphic 3-form and $k$ is the Kähler form on $CY_3$). Let $\eta$ be the volume form on $S^1$ such that $\int_{S^1}\eta = 2\pi R$, then one has the decompositions

$$\begin{aligned}
\phi &= \rho + k\wedge\eta \ , \\
*\phi &= \hat\rho\wedge\eta + \frac{1}{2}k\wedge k \ .
\end{aligned} \tag{A.6}$$

Note that the arbitrary phase $\alpha$ implies that the real/imaginary part of $\Omega$ is not canonically related to $\phi$ or $*\phi$. In the paper we frequently take $\alpha = 0$ but it is possible to have a $CY_3$ sitting in a $G_2$ with a different alignment of its complex structure.

## B. Ghost structure

A special feature of Chern-Simons theory [32] and OSFT [33] (which have similar functional forms) is that it is possible to rewrite the gauge-fixed versions of these theories in the same form as the original theory but with the gauge or string field replaced by an extended field.

We would now like to argue that the higher (in the sense of fermion/ghost number) string modes that are BRST closed can be added to the OSFT action and interpreted as gauge-fixing ghosts or antifields. This was suggested in [13] by noting that the actions of gauge-fixed CS theory [32] and OSFT bear a similar form (this is also discussed directly for physical OSFT in [33]). The main point will be to re-write gauge-fixed CS theory in terms of a 'vector superfield', where the rest of the multiplet comes from the ghosts and antifields. This superfield has the same expansion as the string field $\mathcal{A}(X^\mu, \psi^\mu)$ with the $\psi_0^\mu$'s being replaced by fermionic coordinates. In [32] it is shown that gauge-fixed CS theory written in terms of a superfield like this has exactly the same action as standard CS theory but with $A_\mu \to \mathcal{A}$. This allows us to re-interpret (6.3) with all the terms in the string field as a gauge-fixed version of OSFT that reduces to gauge-fixed CS theory in the large $t$ limit (which would be an exact limit in topological string theory).

### B.1 7-cycle theory

The normal modes introduce additional complications in gauge fixing the theory so to avoid these for now we consider first the theory on the entire $G_2$ manifold. We will gauge fix this theory using the Batalin-Vilkovisky (BV) method of quantization rather than the Faddeev-Popov method, which is used in [32], since BV quantization (which is carried out for OSFT in [33]) makes the connection with the OSFT action (with no constraint on the ghost number of the string field) more transparent. This connection between closed/open string field theory and the gauge-fixed Kodaira-Spencer/holomorphic Chern-Simons description of the B-model has been established in section 5 of [21].

#### B.1.1 BV quantization

To exhibit this similarity, let us consider the BV quantization of the classical action (6.30). We will be brief and refer the reader to the lecture notes of Henneaux [49] for more details.

We introduce an anticommuting scalar $c$ corresponding to the BRST ghost from the gauge symmetry of $S_0 = \int_Y *\phi \wedge CS(A)$, with associated BRST transformations

$$\mathfrak{s} A_\mu = D_\mu c \,, \qquad \mathfrak{s} c = \frac{1}{2}[c, c] \,. \tag{B.1}$$

$A_\mu$ and $c$ have BRST ghost numbers 0 and 1 respectively, and we denote $\mathfrak{s}$ as the BRST charge associated with BV quantization (this should not be confused with the BRST charge $Q$ of the worldsheet or OSFT theory). Recall that in the BV formalism, to each field/ghost $\Phi$ one associates an anti-field/ghost $\Phi^*$ in the Poincaré dual representation of the Lorentz group, with opposite Grassmann parity. Thus we introduce an anticommuting antifield $A^{*\,\mu}$ for $A_\mu$ and a commuting antighost $c^*$ for $c$. These have 'anti-ghost number' 1 and 2 respectively, and their BRST transformations are

$$\mathfrak{s} A^{*\,\mu} = \phi^{\mu\nu\rho} F_{\nu\rho} + [A^{*\,\mu}, c] \,, \qquad \mathfrak{s} c^* = D_\mu A^{*\,\mu} + [c^*, c] \,. \tag{B.2}$$

The nilpotent BRST operator $\mathfrak{s}$ acts on a doubly-graded complex of functionals, the cohomology of which, in degree zero (for both gradings) corresponds to gauge-invariant functionals satisfying the equations of motion. This principle yields the specific BRST transformations above (see [49] for more details). The action of $\mathfrak{s}$ itself defines a single grading on this complex, given by the difference of ghost number and antighost number.

An action functional $S$ is defined as the generating function of the BRST symmetry, such that $\mathfrak{s}\mathcal{F} = (S, \mathcal{F})$ for any functional $\mathcal{F}$ of the fields or antifields. The anti-bracket $(-,-)$ is defined as

$$(A, B) = \frac{\delta^r A}{\delta \Phi} \cdot \frac{\delta^l B}{\delta \Phi^*} - \frac{\delta^r A}{\delta \Phi^*} \cdot \frac{\delta^l B}{\delta \Phi} \,. \tag{B.3}$$

Here $\cdot$ denotes the sum over all fields+ghosts in $\Phi$, each contracted with their dual antipartners in $\Phi^*$, the superscripts $r$ and $l$ denote right and left differentiation. In our case $\Phi = (A_\mu, c)$ and $\Phi^* = (A^{*\mu}, c^*)$. Since the functional $S$ generates the BRST symmetry it must satisfy $(S, S) = 0$ so that $\mathfrak{s}^2 = 0$.

These constraints on $S$ allow us to solve for its explicit form, which is given by an expansion in antighost number (the total ghost number of the functional must be zero so the terms must have equal ghost and antighost numbers) of the form

$$S = S_0 + \int \Phi^* \cdot \mathfrak{s}\Phi \,. \tag{B.4}$$

The term $\Phi^* \cdot \mathfrak{s}\Phi$ again denotes a sum over all fields and ghosts in $\Phi$, contracted with their antipartners in $\Phi^*$ (i.e. $A^{*\mu}\mathfrak{s}A_\mu + c^*\mathfrak{s}c$ in our example). In fact this simple form does not hold in general but is correct for actions with an irreducible closed gauge algebra like the one we are considering. For such actions, Faddeev-Popov gauge-fixing actually suffices but the BV approach makes the relationship to OSFT more clear.

Thus in our case the generator of the BRST symmetry is

$$S = \int_Y \phi^{\mu\nu\rho} \operatorname{Tr}\left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho + \phi_{\mu\nu\alpha} A^{*\alpha} \partial_\rho c + \phi_{\mu\nu\alpha} A^{*\alpha} [A_\rho, c] + \frac{1}{2} \phi_{\mu\nu\rho} c^* [c, c] \right) \,. \tag{B.5}$$

It will be clear below that after a relatively trivial field redefinition, the BV quantized CS theory can be identified with the OSFT action with unconstrained ghost number string fields.

It should be noted that this action actually has a larger gauge symmetry than the original action $S_0$. This is a standard feature of BV quantization and is dealt with by restricting the functional to a graded Lagrangian submanifold of the graded symplectic manifold spanned by the fields and antifields (this essentially eliminates the antifield degrees of freedom). In particular one does not sum over this doubled set of fields in the path integral. A convenient way to eliminate $\Phi^*$ in terms of $\Phi$ is via the gauge fermion method whereby one fixes $\Phi^* = \delta\psi/\delta\Phi$ for some choice of fermionic functional $\psi[\Phi]$ of the fields and ghosts only.

**Stationary phase expansion.** Let us not get into the details of gauge-fixing for the full 7-cycle theory above since it will be difficult to evaluate the exact partition function for this non-linear theory in any case. Rather, let us consider the theory in the weak coupling limit where we can restrict to a quadratic expansion about a point of stationary phase.



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The quadratic part of $\int_Y *\phi \wedge CS(A)$, expanded as $A = A^0 + B$ around a solution $A^0$ of the equation of motion $*\phi \wedge F = 0$ is given by

$$S_{cl}[B] = \int_Y *\phi \wedge \mathrm{Tr}(B \wedge DB) \; . \tag{B.6}$$

The BV quantization of this action proceeds as follows. One finds a minimal solution of the master equations takes the form

$$S_{cl}[B] + \int_Y \mathrm{Tr}(B^{*\mu} D_\mu c) \; , \tag{B.7}$$

which is invariant under the nilpotent BRST transformations

$$\mathfrak{s}B_\mu = D_\mu c \; , \quad \mathfrak{s}c = 0 \; , \quad \mathfrak{s}B^{*\mu} = \phi^{\mu\nu\rho} D_\nu B_\rho \; , \tag{B.8}$$

involving BRST ghost $c$ and antifield $B^{*\mu}$. Since $c$ is now BRST-invariant, $\mathfrak{s}c^*$ can be a general BRST-invariant function. A convenient choice of gauge fermion here is

$$\psi = \int_Y \mathrm{Tr}(\bar{c} D^\mu B_\mu) \; , \tag{B.9}$$

in terms of an additional fermionic scalar $\bar{c}$ that is related to a BRST-trivial bosonic scalar $\varphi$ by $\mathfrak{s}\bar{c} = \varphi$. These fields constitute a non-minimal BRST-invariant addition to the action of the form $\int_Y \mathrm{Tr}(\bar{c}^* \varphi)$, which still solves the master equation (the antifields for $\bar{c}$ and $\varphi$ also form a BRST-trivial pair). Eliminating the antifields via the aforementioned constraint $\Phi^* = \delta\psi/\delta\Phi$ fixes $B^{*\mu} = -D^\mu \bar{c}$, $c^* = 0$, $\bar{c}^* = D^\mu B_\mu$ and $\varphi^* = 0$. Thus the gauge-fixed action takes the familiar form

$$\int_Y *\phi \wedge \mathrm{Tr}\left(B \wedge DB\right) + \mathrm{Tr}\left(\varphi D^\mu B_\mu + \bar{c} D^\mu D_\mu c\right) \; , \tag{B.10}$$

with $\varphi$ acting as Lagrange multiplier imposing the gauge-fixing constraint in the action while $\bar{c}$, $c$ correspond to the fermions that appear in the standard Faddeev-Popov determinant.

### B.1.2 Unconstrained OSFT

Let us now consider the form of the OSFT action if we remove the constraint that the string field must be of ghost number one only. Again, we consider the theory for a brane wrapping the entire $G_2$ manifold and extend the results of section 6 (see e.g. equation (6.19))

$$\begin{aligned} S_{(7)} &= \int_Y \mathcal{A} \star Q\mathcal{A} + \frac{2}{3}\mathcal{A} \star \mathcal{A} \star \mathcal{A} \\ &= \int_Y \phi^{\mu\nu\rho}\, \mathrm{Tr}\left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho + \beta_{\mu\nu}\partial_\rho f + \beta_{\mu\nu}[A_\rho, f] + \frac{1}{2} C_{\mu\nu\rho}\{f,f\}\right) \\ &= \int_Y *\phi \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A + \beta \wedge Df + \frac{1}{2} C\{f,f\}\right) \; , \end{aligned} \tag{B.11}$$

where we have used the full expansion of the string field

$$\mathcal{A}(X^\mu, \psi^\mu) = f(X_0) + A_\mu(X_0)\psi_0^\mu + \beta_{\mu\nu}(X_0)\psi_0^\mu\psi_0^\nu + C_{\mu\nu\rho}(X_0)\psi_0^\mu\psi_0^\nu\psi_0^\rho \; . \tag{B.12}$$

Here the fields $f \in \Lambda^0_{\mathbf{1}}$, $\beta \in \Lambda^2_{\mathbf{7}}$ and $C \in \Lambda^3_{\mathbf{1}}$ are respectively the degree zero, two and three modes of the string field $\mathcal{A}$ in the adjoint representation of the gauge group and $D = d + A$ is the gauge-covariant derivative. The higher degree modes can be redefined in terms of these lower degree ones via Hodge duality.

The interesting feature of (B.11) is that it has the same form as the action (B.5) generated by BV quantizing the associative Chern-Simons action (6.30), corresponding to the ghost number one part of the action. Specifically, the antifield $A^{*\mu}$ in (B.5) is identified with the one-form $\phi^{\mu\nu\rho}\beta_{\nu\rho}$, the antighost $c^*$ is identified with the zero-form $\phi^{\mu\nu\rho}C_{\mu\nu\rho}$ and the ghost $c$ with $f$.

Note that this identification implies the BV commutator $*A^* \wedge [A, c]$ must correspond to the OSFT anticommutator $A \wedge \{f, \beta\}$. As we will explain shortly, this comes about as a result of the different statistics of $(A^{*\mu}, c)$ and $(\beta_{\mu\nu}, f)$ (which pair of fields appear in the (anti-)commutator is not so relevant because the cyclicity of the trace can be used to change them around).

One can check that the linearized equations of motion for $\beta$ and $f$

$$\begin{aligned} \phi^{\mu\nu\rho}\partial_\rho f(X) &= 0 \,, \\ \phi^{\mu\nu\rho}\partial_\rho \beta_{\mu\nu}(X) &= 0 \,, \end{aligned} \tag{B.13}$$

reproduce the linearized $Q$-closure constraint. As with the ghost number one part of the string field, this provides a worldsheet check of the kinetic terms in the OSFT action.

**Ghost correlators.** To check that (B.11) is indeed the correct gauge-fixed OSFT action, or even effective $D$-brane action, let us calculate the correlator of the $\beta \wedge A \wedge f$ term on the disc (or upper half-plane) using the arguments in section 5.1. We will compare this with the expression for the 3-pt vertex $*A^* \wedge [A, c]$ in (B.5). The subtlety that emerges is that the string correlator will involve an anticommutator of Grassmann-even fields while the CS 3-pt function can be recast into a form including a commutator of Grassmann-odd fields. This will offset the fact that $\beta$ and $f$ have different statistics than $A^*$ and $c$.

The twisted correlator

$$\left\langle \beta^i_{\mu\nu}(X)\psi^\mu\psi^\nu A^j_\rho(X)\psi^\rho f^k(X)\right\rangle \,, \tag{B.14}$$

receives contributions from the two inequivalent orderings of the operators on the disk. The $X$ and $\psi$ CFTs can be treated separately and indeed the $X$ CFT reduces to an integral over the $G_2$ manifold (this argument is identical to the 3-pt function calculation in [8]). Using the $\mathrm{SL}(2,\mathbb{R})$ of the upper half-plane and the cyclicity of the trace (the above correlator automatically involves a trace over the lie algebra indices by standard arguments) all possible contributions will be of the form

$$(\beta^i_{\mu\nu}t_i A^j_\rho t_j f^k t_k + A^j_\mu t_j \beta^i_{\nu\rho} t_i f^k t_k)\psi^\mu\psi^\nu\psi^\rho \,. \tag{B.15}$$

Here the $t_i$ are a canonically normalized basis of the lie algebra. As with the $\langle AAA\rangle$ correlator, the worldsheet fermions will be contracted with fermions from $\phi_{\alpha\beta\gamma}\psi^\alpha\psi^\beta\psi^\gamma(z)$



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and all the contractions will actually be equal to each other because the antisymmetry of the fermions cancels against that of $\phi$ so the result is some multiple of

$$\phi^{\mu\nu\rho}\mathrm{Tr}[\beta^i_{\mu\nu}t_iA^j_\rho t_jf^kt_k+A^j_\mu t_j\beta^i_{\nu\rho}t_if^kt_k]=\phi^{\mu\nu\rho}\mathrm{Tr}[A^j_\rho t_jf^kt_k\beta^i_{\mu\nu}t_i+A^j_\mu t_j\beta^i_{\nu\rho}t_if^kt_k]\ . \quad (B.16)$$

Finally this becomes

$$\phi^{\mu\nu\rho}A^j_\mu f^k\beta^i_{\nu\rho}\mathrm{Tr}[t_j\{t_k,t_i\}]=*\phi\wedge\mathrm{Tr}\left(A\wedge\{f,\beta\}\right)\ . \quad (B.17)$$

Let us now compare this to $*A^*\wedge[A,c]$, using $\mathcal{C}_{\mu\nu}=\phi_{\mu\nu\alpha}A^{*\alpha}$ for notational convenience

$$\begin{aligned}\phi^{\mu\nu\rho}\mathcal{C}^i_{\mu\nu}A^j_\rho c^k\mathrm{Tr}[t_it_jt_k-t_it_kt_j]&=\phi^{\mu\nu\rho}A^j_\rho\mathcal{C}^i_{\mu\nu}c^k\mathrm{Tr}[t_jt_kt_i-t_jt_it_k]\\&=\phi^{\mu\nu\rho}\mathrm{Tr}[-A^j_\rho t_jc^kt_k\mathcal{C}^i_{\mu\nu}t_i-A^j_\rho t_j\mathcal{C}^i_{\mu\nu}t_ic^kt_k]\ ,\end{aligned} \quad (B.18)$$

where we have simply used the cyclicity of the trace for the first equality and the Grassmann-odd nature of the coefficients $\mathcal{C}^i$ and $c^k$ for the second. This then becomes

$$-*\phi\wedge\mathrm{Tr}\left(A\wedge\{c,\mathcal{C}\}\right)\ . \quad (B.19)$$

Although this seems like a trivial re-writing it is intended to account for the fact that the statistics of the two fields are different. Indeed we should perhaps have mapped gauge-fixed CS theory to string field theory via $\varepsilon\mathcal{C}\to\beta$ and $\varepsilon c\to f$ with $\varepsilon$ some fixed grassmann-odd variable.

### B.2 Gauge-fixed OSFT action on calibrated cycles and the BV formalism

We now consider the form of the OSFT action upon expansion of the string field on calibrated submanifolds of the $G_2$ manifold, and how its structure has a natural interpretation in terms of the BV antifield formalism. Conceptually this is very similar to the 7-cycle theory but with the added complication of normal modes.

#### B.2.1 3-cycle theory

If one considers the expansion of a general string field on an associative cycle of a $G_2$ manifold there are many string modes coming from excitations in the normal directions

$$\mathcal{A}=f+A_a\psi^a_0+\theta_I\psi^I_0+\beta_{ab}\psi^a_0\psi^b_0+\beta_{aI}\psi^a_0\psi^I_0+\beta_{IJ}\psi^I_0\psi^J_0+C_{abc}\psi^a_0\psi^b_0\psi^c_0+\dots\ . \quad (B.20)$$

The dots represent the higher modes with at least one normal index in them. The expansion above includes all purely tangential modes and the lowest two orders of normal modes but there are additional higher degree modes with one or more normal indices which we have not written out.

Including all these contributions, one obtains from OSFT the full gauge theory action on the 3-cycle

$$\begin{aligned}S_{(3)}=\int_M&\epsilon^{abc}\mathrm{Tr}\left(A_a\partial_bA_c+\frac{2}{3}A_aA_bA_c+\beta_{ab}D_cf+\frac{1}{2}C_{abc}[f,f]\right)\\&+\phi^{aIJ}\mathrm{Tr}\left(\theta_ID_a\theta_J+2\,\beta_{aI}[\theta_J,f]\right)\ ,\end{aligned} \quad (B.21)$$

where we have used the fact that $\beta \in \Lambda^2_{\mathbf{7}}$ and $C \in \Lambda^3_{\mathbf{1}}$ to derive the identities $2\phi^{abc}\beta_{bc} = \phi^{aIJ}\beta_{IJ}$ and $\frac{2}{7}\phi^{abc}C_{abc} = \phi^{aIJ}C_{aIJ}$ using the $G_2$ projection operators in appendix A. We have rescaled the ghost fields for convenience.

The first line in $S_{(3)}$ can be understood in terms of the BV formalism in exactly the same way we have already described for $S_{(7)}$. That is, $f$ is the BRST ghost associated with the gauge symmetry of $A_a$, the antifield $A^{*\,a}$ is identified with $\epsilon^{abc}\beta_{bc}$ and the antighost $f^*$ is identified with $\epsilon^{abc}C_{abc}$. This would thus lead one to the usual gauge-fixed action for pure Chern-Simons theory, were it not for the normal modes. The second line in $S_{(3)}$ would be decoupled, describing 4 free scalars in the abelian theory. The subtlety this second line introduces in the non-abelian theory is that it makes the normal mode action degenerate. In particular it has non-trivial BRST transformation $\mathfrak{s}\theta_I = [\theta_I, f]$ under $\mathfrak{s}A_a = D_a f$, $\mathfrak{s}f = \frac{1}{2}[f,f]$ which follows naturally from the dimensional reduction of the BRST structure in 7 dimensions. Hence $\theta_I$ must also have a fermionic antifield $\theta^{*\,I}$ which is identified with $\phi^{IJa}\beta_{Ja}$ in $S_{(3)}$. The corresponding nilpotent BRST transformations for these antifields

$$\begin{aligned} \mathfrak{s}A^{*\,a} &= \epsilon^{abc}F_{bc} + \phi^{aIJ}[\theta_I, \theta_J] + [A^{*\,a}, f] \;, \\ \mathfrak{s}\theta^{*\,I} &= 2\,\phi^{IaJ}D_a\theta_J + [\theta^{*\,I}, f] \;, \\ \mathfrak{s}f^* &= D_a A^{*\,a} + [\theta_I, \theta^{*\,I}] + [f^*, f] \;, \end{aligned} \tag{B.22}$$

then generate the BRST symmetry of $S_{(3)}$ via the master equation.

The quadratic term in the classical part of $S_{(3)}$, expanded as $A_a = A^0_a + B_a$, $\theta_I = \theta^0_I + \xi_I$ around a solution $(A^0_a, \theta^0_I)$ of the equations of motion is given by

$$S_{cl}[B,\xi] = \int_M \epsilon^{abc}\mathrm{Tr}(B_a D_b B_c) + \phi^{aIJ}\mathrm{Tr}(\xi_I D_a \xi_J + \theta^0_I[B_a, \xi_J]) \;. \tag{B.23}$$

One finds a minimal solution of the master equations for this classical action takes the form

$$S_{cl}[B,\xi] + \int_M \mathrm{Tr}(B^{*\,a}D_a c + \xi^{*\,I}[\theta^0_I, c]) \;, \tag{B.24}$$

which is invariant under the nilpotent BRST transformations

$$\begin{aligned} &\mathfrak{s}B_a = D_a c \;, \qquad \mathfrak{s}\xi_I = [\theta^0_I, c] \;, \qquad \mathfrak{s}c = 0 \;, \\ &\mathfrak{s}B^{*\,a} = \epsilon^{abc}D_b B_c + \phi^{aIJ}[\theta^0_I, \xi_J] \;, \qquad \mathfrak{s}\xi^{*\,I} = \phi^{IaJ}(D_a\xi_J - [\theta^0_J, B_a]) \;, \end{aligned} \tag{B.25}$$

involving BRST ghost $c$ and antifields $B^{*\,a}$, $\xi^{*\,I}$.

Thus far we have not seen any difference in structure to that one would get from dimensional reduction of the 7-dimensional theory. To highlight a potential difference between this reduction and the quantization of the 3-dimensional theory, considered in its own right, we make the choice of gauge fermion

$$\psi = \int_M \mathrm{Tr}(\bar{c}(D_a B^a + \alpha\,[\theta^0_I, \xi^I])) \;, \tag{B.26}$$

involving a constant $\alpha$. One has $\alpha = 1$ from the 7-dimensional perspective but $\alpha = 0$ is a more natural choice in 3 dimensions. The additional fermionic scalar $\bar{c}$ is related to a

BRST-trivial bosonic scalar $\varphi$ by $\mathfrak{s}\bar{c} = \varphi$ as before, and again gives a non-minimal addition to the action $\int_M \mathrm{Tr}(\bar{c}^*\varphi)$.

Eliminating the antifields via $\Phi^* = \delta\psi/\delta\Phi$ fixes $B^{*\,a} = -D^a\bar{c}$, $\xi^{*\,I} = -\alpha\,[\theta_I^0, \bar{c}]$, $c^* = 0$, $\bar{c}^* = D_a B^a + \alpha\,[\theta_I^0, \xi^I]$ and $\varphi^* = 0$. Thus the gauge-fixed action takes the form

$$S_{cl}[B,\xi] + \int_M \mathrm{Tr}\left(\varphi(D_a B^a + \alpha\,[\theta_I^0, \xi^I]) + \bar{c}\left(D_a D^a c + \alpha\,[\theta_I^0, [\theta^{0\,I}, c]]\right)\right) . \tag{B.27}$$

One can check that the $\alpha$-dependent terms combine to form a BRST-exact contribution to this action. Thus we argue that any choice of $\alpha$ will give an equivalent description of the quantum theory and we will take $\alpha = 0$.

### B.2.2 4-cycle theory

A similar expansion of the string field on a coassociative 4-cycle in the $G_2$ manifold gives rise to the full gauge theory action

$$\begin{aligned} S_{(4)} = \int_M & \phi^{Iab}\mathrm{Tr}\big(\theta_I F_{ab}\big) + \frac{2}{3}\,\phi^{IJK}\mathrm{Tr}\big(\theta_I\theta_J\theta_K\big) + \frac{1}{2}\phi^{IJK}\mathrm{Tr}\big(C_{IJK}[f,f]\big) \\ & + 2\,\phi^{Iab}\mathrm{Tr}\big(\beta_{Ia}D_b f\big) + \phi^{IJK}\mathrm{Tr}\big(\beta_{IJ}[\theta_K, f]\big) , \end{aligned} \tag{B.28}$$

where $\beta \in \Lambda^2_{\mathbf{7}}$ and $C \in \Lambda^3_{\mathbf{1}}$ are again used to derive the identities $2\phi^{IJK}\beta_{JK} = \phi^{Iab}\beta_{ab}$, $\frac{2}{7}\phi^{IJK}C_{IJK} = \phi^{Iab}C_{Iab}$ for the ghosts on the 4-cycle.

In terms of the BV formalism, $f$ is again the BRST ghost associated with the gauge symmetry of $A_a$, the antifields $A^{*\,a}$ and $\theta^{*\,I}$ are respectively identified with $\phi^{abI}\beta_{bI}$ and $\phi^{IJK}\beta_{JK}$ whilst the antighost $f^*$ is identified with $\phi^{IJK}C_{IJK}$. The BRST transformations of these fields are again

$$\begin{aligned} \mathfrak{s}A_a &= D_a f \, , \qquad \mathfrak{s}\theta_I = [\theta_I, f] \, , \qquad \mathfrak{s}f = \frac{1}{2}[f,f] \, , \\ \mathfrak{s}A^{*\,a} &= 2\,\phi^{abI}D_b\theta_I + [A^{*\,a}, f] \, , \\ \mathfrak{s}\theta^{*\,I} &= \phi^{Iab}F_{ab} + \phi^{IJK}[\theta_J, \theta_K] + [\theta^{*\,I}, f] \, , \\ \mathfrak{s}f^* &= D_a A^{*\,a} + [\theta_I, \theta^{*\,I}] + [f^*, f] \, , \end{aligned} \tag{B.29}$$

generating a symmetry of $S_{(4)}$ via the master equation.

The quadratic term in the classical part of $S_{(4)}$, expanded as $A_a = A_a^0 + B_a$, $\theta_I = \theta_I^0 + \xi_I$ around a solution $(A_a^0, \theta_I^0)$ of the equations of motion is given by

$$S_{cl}[B,\xi] = \int_M \phi^{Iab}\mathrm{Tr}\big(\xi_I D_a B_b + \theta_I^0 B_a B_b\big) + \phi^{IJK}\mathrm{Tr}\big(\theta_I^0 \xi_J \xi_K\big) . \tag{B.30}$$

One again finds a minimal solution of the master equations for this classical action takes the form

$$S_{cl}[B,\xi] + \int_M \mathrm{Tr}(B^{*\,a}D_a c + \xi^{*\,I}[\theta_I^0, c]) \, , \tag{B.31}$$

which is invariant under the nilpotent BRST transformations

$$\begin{aligned} \mathfrak{s}B_a &= D_a c \, , & \mathfrak{s}\xi_I &= [\theta_I^0, c] \, , \quad \mathfrak{s}c = 0 \, , \\ \mathfrak{s}B^{*\,a} &= \phi^{abI}\left(D_b\xi_I - [\theta_I^0, B_b]\right) \, , & \mathfrak{s}\xi^{*\,I} &= \phi^{Iab}D_a B_b + \phi^{IJK}[\theta_J^0, \xi_K] \, , \end{aligned} \tag{B.32}$$



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involving BRST ghost $c$ and antifields $B^{*\,a}$, $\xi^{*\,I}$.

We again make the choice of gauge fermion

$$\psi = \int_M \mathrm{Tr}\left(\bar{c}\left(D_a B^a + \alpha\ \left[\theta_I^0, \xi^I\right]\right)\right)\ , \tag{B.33}$$

involving the constant $\alpha$. Everything now follows just as for the 3-cycle case. The gauge-fixed action takes the form

$$S_{cl}[B,\xi] + \int_M \mathrm{Tr}\left(\varphi\left(D_a B^a + \alpha\ \left[\theta_I^0, \xi^I\right]\right) + \bar{c}\left(D_a D^a c + \alpha\ \left[\theta_I^0, \left[\theta^{0\,I}, c\right]\right]\right)\right)\ , \tag{B.34}$$

and we choose $\alpha = 0$ to ignore the BRST-exact $\alpha$-dependent terms.

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